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Geometry of a Family of Quartic Polynomials

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ABSTRACT. For a fixed $\mathcal{A} \in \mathbb{C}$ with $|\mathcal{A}| = 1$, let $\mathcal{P}$ denote the family of complex-valued polynomials of the form $p(z) = (z-1)(z-\mathcal{A})(z-r_1)(z-r_2)$ with $|r_1| = |r_2| = 1$. By the Gauss-Lucas Theorem, the critical points of a polynomial in $\mathcal{P}$ lie in the unit disk. This paper characterizes the location and structure of these critical points. We show that the unit disk contains 'desert' regions in which critical points of polynomials in $\mathcal{P}$ do not occur. In fact, depending on the location of $\mathcal{A}$, the unit disk contains one or two desert regions bounded by the curve implicitly defined by

$$|2z - (\mathcal{A}+1)| = |4z^2 - 3(\mathcal{A}+1)z + 2\mathcal{A}|.$$

In addition to determining where critical points of polynomials in $\mathcal{P}$ are located, we also show that almost every c inside the unit disk and outside the desert region(s) is the critical point of a unique polynomial in $\mathcal{P}$.

1. Introduction

The Gauss-Lucas Theorem (Marden, 1966) guarantees that the critical points of a complex-valued polynomial lie in the convex hull of the roots of that polynomial. Moreover, a critical point lies on the boundary of the convex hull if and only if the critical point is a multiple root of the polynomial. Refinements of the Gauss-Lucas Theorem (see Steinerberger (2020) and the numerous references within) seek to characterize regions which contain all, some, or none of the critical points of a polynomial.

One such refinement (Steinerberger, 2020) shows that if a polynomial p has $m+n$ roots with n roots inside the unit disk and m roots outside the unit disk, then there exists a constant $d_0 > 1$ such that $n-1$ critical points of p lie inside the unit disk and the other m critical points have modulus larger than d_0. That is, there exists an annular region $\{z : 1 < |z| \leq d_0\}$ containing no critical points of p. Another refinement (Rüdinger, 2014) investigates polynomials with a zero inside the convex hull of its roots. Of particular interest is a degree four polynomial with four distinct roots forming a concave quadrilateral. If p is such a polynomial, then one of the three triangles formed by the roots of p contains no critical points of p.

Investigating polynomials whose roots lie on a circle leads to similar refinements of the Gauss-Lucas Theorem. By changing coordinates, such a polynomial can be normalized to have roots on the unit circle with one root located at $z = 1$. By the Gauss-Lucas Theorem, the critical points of such a polynomial must lie in the unit disk. Families of polynomials of this form are investigated in (Frayer et al., 2014; Frayer, 2017; Frayer and Gauthier, 2018), and in each case, the unit disk contains 'desert' regions where critical points do not occur. For a fixed $a \in [-1, 1]$, Frayer and

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Thomson (2020) extends these results to the family of complex-valued polynomials

$$\Omega_a = \{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z-1)(z-a)(z-r_1)(z-r_2), |r_1| = |r_2| = 1\}.$$

For fixed z_1 and z_2 on the unit circle, it follows from (Rüdinger, 2014) and/or (Steinerberger, 2020), depending on the position of a relative to z_1 and z_2, that a region inside the unit disk contains no critical points of $(z-1)(z-a)(z-z_1)(z-z_2) \in \Omega_a$. Surprisingly, as r_1 and r_2 vary around the unit circle, the unit disk contains desert regions which contain no critical points of polynomials in Ω_a. Furthermore, almost every c inside the unit disk and outside the desert regions is the critical point of a unique polynomial in Ω_a.

For a fixed $\mathcal{A} \in \mathbb{C}$ with $|\mathcal{A}| = 1$, this paper extends the ideas of Frayer and Thomson (2020) to the family of quartic polynomials

$$\mathcal{P} = \{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z-1)(z-\mathcal{A})(z-r_1)(z-r_2), |r_1| = |r_2| = 1\}.$$

Once again, the unit disk contains desert regions where critical points of polynomials in $\mathcal{P}$ do not occur. In fact, depending upon the location of $\mathcal{A}$, the unit disk contains one or two desert regions, and almost every c inside the unit disk and outside the desert region(s) is the critical point of a unique polynomial in $\mathcal{P}$ (see our Theorem 6).

2. Critical Points

Suppose $\mathcal{A} \in \mathbb{C}$ with $|\mathcal{A}| = 1$ and $|r_1| = |r_2| = 1$. Then, by the Gauss-Lucas Theorem

$$p(z) = (z-1)(z-\mathcal{A})(z-r_1)(z-r_2) \in \mathcal{P}$$

has three critical points, the zeros of p', that lie in the unit disk.

2.1. Preliminary Information

We begin our analysis of these critical points by introducing a result from Frayer et al. (2014). Given $\alpha > 0$, we let T_α denote the circle of diameter α that passes through 1 and $1-\alpha$ in the complex plane.

Theorem 1. *(Frayer et al., 2014). Let $f(z) = (z-1)(z-z_1)\cdots(z-z_n)$, where $z_k = e^{i\theta_k}$ for each k. Let $c_1, \ldots, c_n$ denote the critical points of $f(z)$, and suppose that $1 \neq c_k \in T_{\alpha_k}$ for each k. Then*

$$\sum_{k=1}^{n} \frac{1}{\alpha_k} = n.$$

Applying Theorem 1 to polynomials in $\mathcal{P}$ gives a disk, $|z - \frac{3}{4}| < \frac{1}{4}$, where critical points cannot occur.

Theorem 2. *No polynomial in $\mathcal{P}$ has a critical point strictly inside $T_{1/2}$.*

Proof. Let c_1, c_2, and c_3 be critical points of $p(z) = (z-1)(z-\mathcal{A})(z-r_1)(z-r_2) \in \mathcal{P}$ with $c_1 \in T_{\alpha_1}$, $c_2 \in T_{\alpha_2}$, and $c_3 \in T_{\alpha_3}$. Theorem 1 implies

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} = 3.$$

Suppose to the contrary that $\alpha_3 < \frac{1}{2}$. Then,

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 3 - \frac{1}{\alpha_3} = \beta < 1$$

implies $\alpha_1 = \alpha_2 = \frac{2}{\beta}$, or $\alpha_1 < \frac{2}{\beta}$ and $\alpha_2 > \frac{2}{\beta}$. Either possibility is a contradiction as $\frac{2}{\beta} > 2$ and all three critical points must lie in the unit disk. □

Similarly, as a consequence of Theorem 2, no polynomial in $\mathcal{P}$ has a critical point inside the disk $\left|z - \frac{3}{4}\mathcal{A}\right| < \frac{1}{4}$.

For $p(z) = (z-1)(z-\mathcal{A})(z-r_1)(z-r_2) \in \mathcal{P}$, rotation about the origin by $\frac{1}{2}\text{Arg}(\mathcal{A})$ radians in the clockwise direction forces the roots located at $\mathcal{A}$ and 1 to become complex conjugates. This symmetry is conducive to further analysis. For a fixed $A \in \mathbb{C}$ with $|A| = 1$ we let P_A denote the family of polynomials

$$P_A = \left\{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z-A)(z-\overline{A})(z-r_1)(z-r_2), |r_1| = |r_2| = 1\right\}.$$

This paper will characterize the critical points of polynomials in P_A (see Theorem 5), and then apply those results to the original family of polynomials $\mathcal{P}$ (see Theorem 6).

As an initial observation, Theorem 2 implies that the unit disk contains two disks in which critical points of polynomials in P_A cannot occur.

Corollary 1. *No polynomial in P_A has a critical point in the open disk $\left|z - \frac{3}{4}A\right| < \frac{1}{4}$ or $\left|z - \frac{3}{4}\overline{A}\right| < \frac{1}{4}$.*

We now investigate several examples. For convenience, let U denote the unit circle.

Example 1. *A polynomial $p \in P_A$ has a critical point at A whenever A is a repeated root of p. So, for each $r \in U$,*

$$(z-A)^2(z-\overline{A})(z-r) \in P_A$$

has a critical point at A. Similarly, for each $r \in U$,

$$(z-A)(z-\overline{A})^2(z-r) \in P_A$$

has a critical point at $\overline{A}$.

As Example 1 describes the only polynomials in P_A with a critical point at A or $\overline{A}$, we will assume that $c \notin \{A, \overline{A}\}$ as necessary through the remainder of the paper. Before exploring another example, we define and analyze an important family of curves.

Definition 1. *Let $A \in U$ with Re$(A) = \alpha$ and define*

$$D_A = \{z \in \mathbb{C} \ : \ \left|2z^2 - 3\alpha z + 1\right| = |z - \alpha|\}.$$

The set D_A depends upon $A \in U$. To visualize D_A, we explain how D_A changes as A moves around the unit circle. If A starts at $z = 1$ and moves around the unit circle in the counterclockwise direction, D_A is a simple closed curve tangent to U at A and $\overline{A}$. When $\text{Re}(A) = \frac{7}{9}$, the curve bifurcates into two disconnected simple closed curves tangent to U at A and $\overline{A}$. See Figure 2.1. By symmetry, as A moves past $z = i$ the two curves come back together when $\text{Re}(A) = -\frac{7}{9}$. For future use we note that both $\frac{3}{4}A$ and $\frac{3}{4}\overline{A}$ are contained inside D_A. A GeoGebra animation illustrating D_A for varying values of A can be found at `https://people.uwplatt.edu/~frayerc/DA.html`.

Example 1 shows that infinitely many polynomials in P_A have a critical point at $c \in \{A, \overline{A}\}$. Similarly, one might wonder how many polynomials in P_A will have a critical point at $r \in U \setminus \{A, \overline{A}\}$.

Example 2. *By the Gauss-Lucas Theorem,*

$$p_r(z) = (z-A)(z-\overline{A})(z-r)^2 \in P_A$$

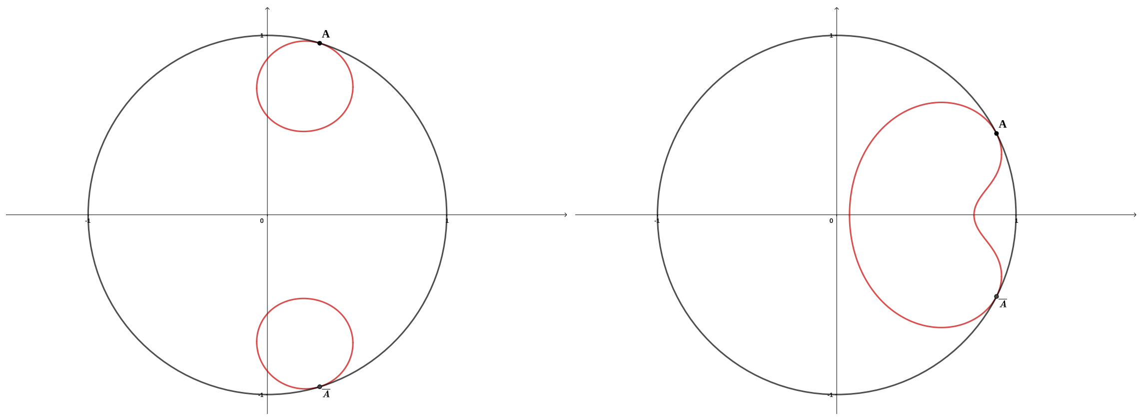

FIGURE 2.1. The curve D_A when $0 < \text{Re}(A) < \frac{7}{9}$ on the left, and when $\frac{7}{9} < \text{Re}(A) < 1$ on the right.

is the only polynomial in P_A with a critical point at $r \in U \setminus \{A, \overline{A}\}$. Differentiating and simplifying gives

$$p_r'(z) = 2(z-r)(2z^2 - (r+3\alpha)z + r\alpha + 1).$$

Thus, p_r has a critical point at $z = r$, as expected, and two other critical points that satisfy $2c^2 - (r+3\alpha)c + r\alpha + 1 = 0$. Rewriting as $2c^2 - 3\alpha c + 1 = r(c-\alpha)$, taking the modulus of both sides, and noting that $r \in U$ yields

$$\left|2c^2 - 3\alpha c + 1\right| = |c - \alpha|\,.$$

To summarize, $p_r(z) = (z-A)(z-\overline{A})(z-r)^2$ is the unique polynomial in P_A with a critical point at $r \in U \setminus \{A, \overline{A}\}$. Furthermore, the other two critical points of p_r lie on the curve D_A.

2.2. The General Case

A polynomial of the form $p(z) = (z-A)(z-\overline{A})(z-r_1)(z-r_2) \in P_A$ has three critical points in the unit disk. To further understand these critical points, we investigate how a critical point of p is related to r_1 and r_2. For $\alpha = \text{Re}(A)$,

$$p(z) = (z^2 - 2\alpha z + 1)(z - r_1)(z - r_2)$$

and differentiation gives

$$p'(z) = 4z^3 - (3r_1 + 3r_2 + 6\alpha)z^2 + (4\alpha r_1 + 4\alpha r_2 + 2r_1r_2 + 2)z - r_1 - r_2 - 2\alpha r_1 r_2.$$

If c is a critical point of $p(z)$, then

$$0 = p'(c) = 4c^3 - (3r_1 + 3r_2 + 6\alpha)c^2 + (4\alpha r_1 + 4\alpha r_2 + 2r_1r_2 + 2)c - r_1 - r_2 - 2\alpha r_1 r_2.$$

Solving for r_1 gives

$$r_1 = \frac{(3c^2 - 4\alpha c + 1)r_2 - (4c^3 - 6\alpha c^2 + 2c)}{(2c - 2\alpha)r_2 - (3c^2 - 4\alpha c + 1)}.$$

Definition 2. *Given $c \in \mathbb{C}$, we define*

$$f_c(z) = \frac{(3c^2 - 4\alpha c + 1)z - (4c^3 - 6\alpha c^2 + 2c)}{(2c - 2\alpha)z - (3c^2 - 4\alpha c + 1)}$$

and let $S_c = f_c(U)$.

Observe that f_c is a Mobius transformation with $f_c(r_2) = r_1$. Furthermore, for $c \in \mathbb{C} \setminus \{A, \overline{A}\}$, $(f_c)^{-1} = f_c$, and it follows that $f_c(r_1) = r_2$. We have established the following result.

Theorem 3. *A polynomial $p(z) = (z - A)(z - \overline{A})(z - r_1)(z - r_2) \in P_A$ has a critical point at c if and only if $f_c(r_2) = r_1$.*

Since $r_1, r_2 \in U$, $f_c(r_1) = r_2 \in S_c$ and $f_c(r_2) = r_1 \in S_c$ implies $\{r_1, r_2\} \subseteq S_c \cap U$. Lemma 1 investigates $|S_c \cap U|$ and is a direct extension of a result from (Frayer and Thomson, 2020).

Lemma 1. *Suppose $c \in \mathbb{C}$.*

(1) If $S_c \cap U = \varnothing$, then no polynomial in P_A has a critical point at c.
(2) If $S_c = U$, then infinitely many polynomials in P_A have a critical point at c.
(3) If $|S_c \cap U| \in \{1, 2\}$, then c is the critical point of a unique polynomial in P_A.

As $S_c = f_c(U)$, characterizing the critical points of polynomials in P_A requires a better understanding of the Mobius transformation f_c. Direct computations show that the Mobius transformation f_c has two fixed points, c and $q = \dfrac{2c^2 - 3\alpha c + 1}{c - \alpha}$, and pole $z_\infty = \dfrac{3c^2 - 4\alpha c + 1}{2c - 2\alpha}$ (also the pole of inversion) which is the midpoint of the line segment connecting c and q. The normal form of the Mobius transformation is given by

$$\frac{f_c(z) - c}{f_c(z) - q} = re^{i\theta} \frac{z - c}{z - q}.$$

Observing that $f_c(z_\infty) = \infty$ and manipulating algebraically gives $re^{i\theta} = -1$. Therefore, f_c is an elliptic Mobius transformation. See (Hitchman, 2018).

Elliptic Mobius transformations can be expressed as a composition of two inversions about clines (circles or lines). In this case, as z_∞ is on the line L passing through c and q, for Ω the circle centered at z_∞ passing through c and q,

$$f_c(z) = i_\Omega(r_L(z))$$

where i_Ω is inversion about the circle Ω and r_L is reflection about the line L. This allows us to visualize $S_c = f_c(U)$ geometrically and will be useful for future observations. See Figure 2.2.

3. Properties of S_c

Suppose $c \notin \{A, \overline{A}\}$. Since f_c is a Mobius transformation and U is a circle, $S_c = f_c(U)$ is a circle or a line. To further understand S_c, we begin with a special case. To determine the values of c for which $S_c = U$, we make use of the following result.

Theorem 4. *(see Frayer, 2017, Theorem 2) A Mobius transformation T sends the unit circle to the unit circle if and only if*

$$T(z) = \frac{\overline{\alpha} z - \overline{\beta}}{\beta z - \alpha}$$

for some $\alpha, \beta \in \mathbb{C}$ with $|\frac{\alpha}{\beta}| \neq 1$.

Applying Theorem 4 to

$$f_c(z) = \frac{(3c^2 - 4\alpha c + 1)z - (4c^3 - 6\alpha c^2 + 2c)}{(2c - 2\alpha)z - (3c^2 - 4\alpha c + 1)}$$

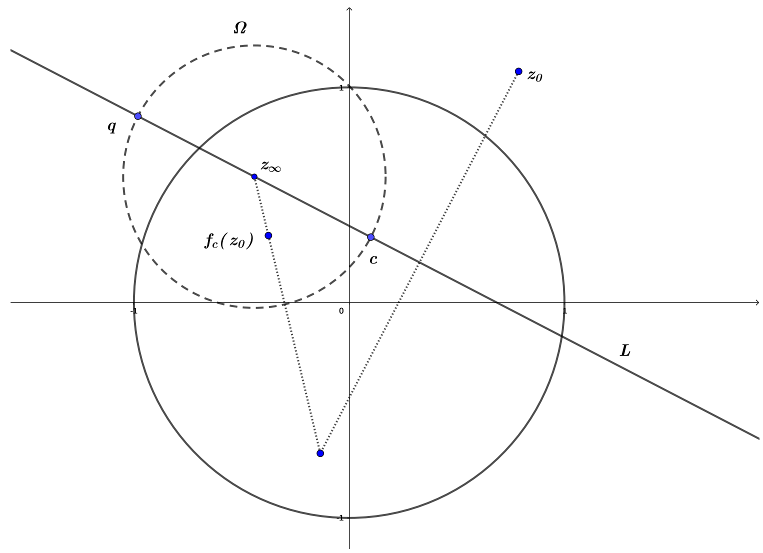

FIGURE 2.2. The Mobius transformation f_c can be visualized as the composition of two inversions: reflection about the line L and inversion about the circle Ω.

implies that $S_c = U$ whenever c satisfies

$$\overline{3c^2 - 4\alpha c + 1} = 3c^2 - 4\alpha + 1 \quad \text{and} \quad \overline{2c - 2\alpha} = 4c^3 - 6\alpha c^2 + 2c. \tag{3.1}$$

Manipulating the left equation in (3.1) and setting $c = x + iy$ gives

$$\begin{aligned} 3\overline{c}^2 - 4\alpha\overline{c} + 1 &= 3c^2 - 4\alpha c + 1 \\ (\overline{c} - c)(6x - 4\alpha) &= 0 \\ (-2yi)(6x - 4\alpha) &= 0. \end{aligned}$$

Therefore $c = x + iy$ satisfies the left equation in (3.1) whenever $y = 0$ or $x = \frac{2}{3}\alpha$. Substituting $c = \frac{2}{3}\alpha + iy$ into the right equation in (3.1) and equating real and imaginary parts eventually gives

$$\begin{aligned} \text{Re}: &\quad 2y^2 + \frac{40}{27}\alpha^2 - 2 = 0 \\ \text{Im}: &\quad 4y^3 + \frac{24}{9}\alpha^2 y - 4y = 0. \end{aligned}$$

Since this system of equations has no solution, $c = \frac{2}{3}\alpha + iy$ does not satisfy (3.1), and the only remaining possibility is $y = 0$. Substituting $c = x$ into the right equation in (3.1) gives

$$4x^3 - 6\alpha x^2 + 2\alpha = 0. \tag{3.2}$$

For $\alpha \in (-1, 1)$, (3.2) has a unique solution, call it c_∞, with $c_\infty \in (-1, 1)$. Furthermore, when $\alpha = 1$ we have $A = \overline{A} = 1$ and (3.2) has two solutions: $-\frac{1}{2}$ and 1. As Example 1 characterized the polynomials in P_A with a critical point at A (or $\overline{A}$) we only have one solution of interest, $c_\infty = -\frac{1}{2}$. Similarly, when $\alpha = -1$ we have $A = \overline{A} = -1$ and $c_\infty = \frac{1}{2}$. We have established the following result.

Lemma 2. *Suppose $A \in U$ with Re$(A) = \alpha$ and c_∞ the unique value in $(-1, 1)$ with $4(c_\infty)^3 - 6\alpha(c_\infty)^2 + 2\alpha = 0$. Then, $S_c = U$ if and only if $c = c_\infty$.*

As another special case, note that S_c is a line whenever there exists a $z_0 \in U$ with

$$(2c - 2\alpha)z_0 - (3c^2 - 4\alpha c + 1) = 0.$$

Adding the right term to both sides, taking the modulus, and noting that $|z_0| = 1$ implies S_c is a line if and only if

$$|2c - 2\alpha| = |3c^2 - 4\alpha c + 1|. \tag{3.3}$$

For $|2c-2\alpha| \neq |3c^2-4\alpha c+1|$, S_c is a circle. By the definition of S_c, $z \in S_c$ if and only if there exists some $w \in U$ with $f_c(w) = z$. Equivalently, $f_c^{-1}(z) = f_c(z) = w$ implies $|f_c(z)| = |w| = 1$, and so

$$\left|\frac{(3c^2 - 4\alpha c + 1)z - (4c^3 - 6\alpha c^2 + 2c)}{(2c - 2\alpha)z - (3c^2 - 4\alpha c + 1)}\right| = 1.$$

Therefore, $z \in S_c$ if and only if

$$\left|z - \frac{3c^2 - 4\alpha c + 1}{2c - 2\alpha}\right| = \left|\frac{3c^2 - 4\alpha c + 1}{2c - 2\alpha}\right| \left|z - \frac{4c^3 - 6\alpha c^2 + 2c}{3c^2 - 4\alpha c + 1}\right|. \tag{3.4}$$

When $d \neq 1$, the solution set of

$$|z - u| = d|z - v|$$

is a circle of Appollonius (see Partenskii (2008)) and has center C and radius R satisfying

$$C = \frac{d^2 v - u}{d^2 - 1} \quad \text{and} \quad R = |v - u| \left|\frac{d}{d^2 - 1}\right|. \tag{3.5}$$

When $d = \left|\frac{3c^2-4\alpha c+1}{2c-2\alpha}\right| = 1$ in (3.4), S_c is a line (verifying our previous observation in equation (3.3)). When $d \neq 1$, S_c is a circle with center

$$C = \frac{\left|\frac{3c^2-4\alpha c+1}{2c-2\alpha}\right|^2 \frac{4c^3-6\alpha c^2+2c}{3c^2-4\alpha c+1} - \frac{3c^2-4\alpha c+1}{2c-2\alpha}}{\left|\frac{3c^2-4\alpha c+1}{2c-2\alpha}\right|^2 - 1}. \tag{3.6}$$

As one last case of interest, we determine when S_c is tangent to U. According to Lemma 1 and Example 2, if S_c is tangent to U at $r \notin \{A, \overline{A}\}$, then c is a critical point of the unique polynomial

$$p_r(z) = (z - A)(z - \overline{A})(z - r)^2 \in P_A.$$

Furthermore, as seen in Example 2, the three critical points of p_r satisfy $c_1 = r$ and $c_{2,3} \in D_A$. Therefore, if S_c is tangent to U, then $c \in U \cup D_A$. Let's explore S_c when $c \in U \cup D_A$.

Lemma 3. *If $c \in D_A \setminus \{A, \overline{A}\}$, then S_c is internally tangent to U at* $\dfrac{2c^2 - 3\alpha c + 1}{c - \alpha}$.

Proof. Let $c \in D_A$. Then $|2c^2 - 3\alpha c + 1| = |c - \alpha|$ and it follows that

$$\left|\frac{2c^2 - 3\alpha c + 1}{c - \alpha}\right| = 1.$$

Letting $e^{i\phi} = \dfrac{2c^2 - 3\alpha c + 1}{c - \alpha}$, direct calculations show that $f_c(e^{i\phi}) = e^{i\phi} \in S_c$. Furthermore, using standard manipulations and the definition of D_A, the center of S_c becomes

$$\begin{aligned} C &= \frac{\overline{\frac{3c^2-4\alpha c+1}{2c-2\alpha}}\,\frac{4c^3-6\alpha c^2+2c}{2c-2\alpha} - \frac{3c^2-4\alpha c+1}{2c^2-3\alpha c+1}\,\frac{2c^2-3\alpha c+1}{2c-2\alpha}\,\frac{2\bar{c}^2-3\alpha\bar{c}+1}{2\bar{c}^2-3\alpha\bar{c}+1}}{\left|\frac{3c^2-4\alpha c+1}{2c-2\alpha}\right|^2 - 1} \\ &= \frac{(\overline{3c^2 - 4\alpha c + 1})(2c - 2\alpha)ce^{i\phi} - (3c^2 - 4\alpha c + 1)(\overline{2c^2 - 3\alpha c + 1})2e^{i\phi}}{|3c^2 - 4\alpha c + 1|^2 - |2c - 2\alpha|^2} \\ &= \frac{C_N}{C_D}e^{i\phi}. \end{aligned}$$

Observing that $C_D \in \mathbb{R}$ and further simplifying C_N eventually gives $C = Ke^{i\phi}$ with $K \in \mathbb{R}$. Then, as $e^{i\phi} \in S_C \cap U$ is on the line segment connecting the centers of S_c and U, S_c is tangent to U at $e^{i\phi}$. It remains to show that S_c is internally tangent to U.

To show that S_c is internally tangent to U, we verify that there exists a $z_0 \in U$ with $|f_c(z_0)| < 1$. We do so by recalling the geometric interpretation of f_c. Letting L represent the line through c and $e^{i\phi}$ (the two fixed points of f_c) and Ω the circle with diameter passing through c and $e^{i\phi}$, $f_c(z) = i_\Omega(r_L(z))$ where i_Ω is inversion about the circle Ω and r_L is reflection about the line L. See Figure 3.1. If E represents the second point of intersection between L and U, then $f_c(E) = i_\Omega(r_L(E)) = i_\Omega(E) \in S_c$. Furthermore, as E is outside of Ω, $f_c(E) = i_\Omega(E)$ is inside Ω with $|f_c(E)| < 1$. Therefore S_c is internally tangent to U at $e^{i\phi}$. □

Similar, but less involved computations show that when $c \in U$, S_c is externally tangent to U at c with $C = \frac{4}{3}c$. We have established the following result.

Lemma 4. *Suppose $c \in \mathbb{C} \setminus \{A, \overline{A}\}$.*

(1) S_c is internally tangent to U if and only if $c \in D_A$.
(2) S_c is externally tangent to U if and only if $c \in U$.

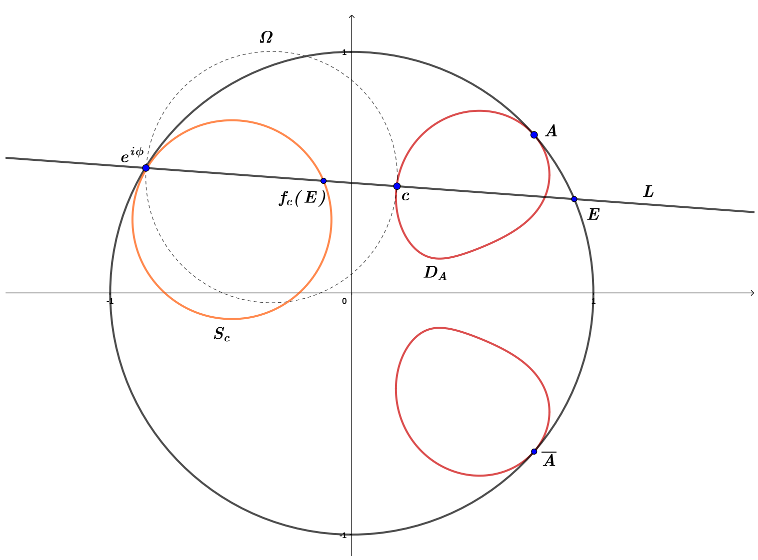

FIGURE 3.1. When $c \in D_A$, $e^{i\phi} \in U$ and $E \in L \cap U$ is outside of Ω. Therefore, $f_c(E) = i_\Omega(r_L(E)) = i_\Omega(E) \in S_c$ with $|f_c(E)| < 1$.

4. Main Results

We let $\mathcal{O}_A$ represent the open region inside the unit disk and outside the region(s) bounded by D_A. Visually, in Figure 2.1, $\mathcal{O}_A$ is the region inside, but not on, the unit circle and enclosed between, but not on, the D_A curves. Denote the closure of $\mathcal{O}_A$ by $\overline{\mathcal{O}_A}$.

When $c \in D_A \cup U$, S_c is tangent to U. Lemmas 5 and 6 determine $|S_c \cap U|$ when $c \notin D_A \cup U$.

Lemma 5. *If c is contained inside D_A, then $S_c \cap U = \varnothing$.*

Proof. Let c be contained inside D_A and recall that $\frac{3}{4}A$ and $\frac{3}{4}\overline{A}$ are also contained inside D_A. As $c \notin D_A \cup U$, S_c is not tangent to U. Suppose to the contrary that $|S_c \cap U| = 2$. As we drag c to $\frac{3A}{4}$ or $\frac{3\overline{A}}{4}$ (which ever is closer) along a line segment contained in D_A, S_c is continuously transformed into a circle not intersecting U. By the Intermediate Value Theorem, there must exist a c_0 on the line segment with S_{c_0} tangent to U. However, as the line segment does not intersect $D_A \cup U$, this contradicts Lemma 4 and it follows that $S_c \cap U = \varnothing$.

□

A similar argument can be used to prove Lemma 6.

Lemma 6. *If $c \in \mathcal{O}_A \setminus \{c_\infty\}$, then $|S_c \cap U| = 2$.*

We are now ready to characterize the critical points of polynomials in P_A.

Theorem 5. *Let $c \in \mathbb{C}$.*

(1) If $c \notin \overline{\mathcal{O}_A}$, then no polynomial in P_A has a critical point at c.
(2) If $c \in \{c_\infty, A, \overline{A}\}$, then infinitely many polynomials in P_A have a critical point at c.
(3) If $c \in \overline{\mathcal{O}_A} \setminus \{c_\infty, A, \overline{A}\}$, then a unique polynomial in P_A has a critical point at c.

Proof. Let $c \in \mathbb{C}$.

(1) If c is inside D_A, Lemmas 5 and 1 imply that no polynomial in P_A has a critical point at c. Furthermore, by the Gauss-Lucas Theorem, no polynomial in P_A has a critical point outside the unit disk.
(2) If $c = c_\infty$, then Lemma 2 implies $S_c = U$. Therefore, by Lemma 1 and Theorem 3,
$$(z - A)(z - \overline{A})(z - r)(z - f_c(r)) \in P_A$$
has a critical point at c for each $r \in U$. By Example 1, there are infinitely many polynomials in P_A with a critical point at $c \in \{A, \overline{A}\}$.
(3) If $c \in \overline{\mathcal{O}_A} \setminus \{c_\infty, A, \overline{A}\}$, $c \in \mathcal{O}_A \setminus \{c_\infty\}$ or $c \in \{U \cup D_A\} \setminus \{A, \overline{A}\}$. When $c \in \mathcal{O}_A \setminus \{c_\infty\}$ Lemma 6 implies $|S_c \cap U| = 2$. When $c \in \{U \cup D_A\} \setminus \{A, \overline{A}\}$ Lemma 4 implies that $|S_c \cap U| = 1$. Therefore, by Lemma 1, there is a unique polynomial in P_A with a critical point at c

□

We finish our discussion by revisiting the family of polynomials $\mathcal{P}$. For a fixed $A \in U$ and some $r_1, r_2 \in U$,
$$(z - A)(z - \overline{A})(z - r_1)(z - r_2) \in P_A.$$
Counterclockwise rotation by $\mathrm{Arg}(A)$ gives
$$(z - A^2)(z - 1)(z - \widetilde{r_1})(z - \widetilde{r_2}) \in \mathcal{P}$$

and demonstrates a one-to-one correspondence between P_A and $\mathcal{P}$. In fact, by redefining D_A, $\mathcal{O}_A$ and c_∞, Theorem 5 can easily be restated for polynomials in $\mathcal{P}$.

For $\mathcal{A} \in U$, we let D denote the set of complex numbers satisfying

$$|2z - (\mathcal{A}+1)| = |4z^2 - 3(\mathcal{A}+1)z + 2\mathcal{A}|$$

and $\mathcal{O}$ represent the open region enclosed inside the unit disk and outside the region(s) bounded by D. To visualize D and $\mathcal{O}$, rotate the images in Figure 2.1 so that $\overline{A}$ becomes $z = 1$. Furthermore, for $\alpha = \text{Re}(\mathcal{A})$ and k the unique real solution of $8(\alpha+1)k^3 - 6(\alpha+1)k^2 + 1 = 0$, we define $\widetilde{c_\infty} = k(\mathcal{A}+1)$. We are now able to restate Theorem 5 for polynomials in $\mathcal{P}$.

Theorem 6. *Let* $c \in \mathbb{C}$.

(1) If $c \notin \overline{\mathcal{O}}$*, then no* $p \in \mathcal{P}$ *has a critical point at* c.
(2) If $c \in \{\widetilde{c_\infty}, \mathcal{A}, 1\}$*, then infinitely many polynomials in* $\mathcal{P}$ *have a critical point at* c.
(3) If $c \in \overline{O} \setminus \{\widetilde{c_\infty}, \mathcal{A}, 1\}$*, then a unique polynomial in* $\mathcal{P}$ *has a critical point at* c.

This completes our analysis of critical points of polynomials in $\mathcal{P}$, and as usual, many interesting questions remain. It would be nice to characterize critical points of families of polynomials similar to $\mathcal{P}$ and Ω_a. For example, when $A \in \mathbb{C} \setminus \mathbb{R}$ with $|A| < 1$, what can be said about critical points of polynomials of the form

$$p(z) = (z-1)(z-A)(z-r_1)(z-r_2)$$

with $|r_1| = |r_2| = 1$? Such a polynomial lies 'inbetween' $\mathcal{P}$ and Ω_a. Preliminary analysis suggests that our methods apply nicely to this scenario. For fixed z_1 and z_2 on the unit circle, it follows from (Rüdinger, 2014) and/or (Steinerberger, 2020), depending on the position of A relative to z_1 and z_2, that a region inside the unit disk contains no critical points of $(z-1)(z-A)(z-z_1)(z-z_2)$. But what more can we say about the existence of a desert region(s) as r_1 and r_2 vary around the unit circle? Furthermore, for a specified $c \in \mathbb{C}$, how many, if any, such polynomials will have a critical point at c? Much more is waiting to be investigated.

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