North Carolina Journal of
Mathematics and Statistics
Volume 1, Pages 22–29 (Accepted August 24, 2015, published October 1, 2015)
ISSN 2380-7539

A Discrete Resonance Problem with Periodic Nonlinear Forcing

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ABSTRACT. In this paper, we study the problem $Ax = \lambda_1 x - cv(x), x \in \mathbb{R}^n$ where A is a Laplacian matrix, λ_1 is the principal eigenvalue of A, $c \in \mathbb{R}$, $v(x)$ is a periodic gradient vector field, and $n = 2$ or 3. We show that the problem has infinitely many solutions with all positive components and infinitely many solutions with all negative components. With certain restrictions on c we provide more detailed information about the solution set. In particular, for the $n = 3$ case we prove that the solutions can be characterized as local minima and saddle points of the associated functional.

1. Introduction

In this paper, we study the two discrete problems

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - c \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \end{bmatrix}, \tag{1.1}$$

and

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (2 - \sqrt{2}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - c \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \\ \sin(x_3) \end{bmatrix}. \tag{1.2}$$

These problems are both of the form

$$Ax = \lambda_1 x - cv(x), x \in \mathbb{R}^n \tag{1.3}$$

where A is a Laplacian matrix, λ_1 is the principal eigenvalue of A, $c \in \mathbb{R}$, and $v(x)$ is a periodic gradient vector field. We note that λ_1 is positive and simple, the corresponding eigenvector, e_1, has all positive components, and any other eigenvalue, λ, of A is a real number satisfying $\lambda > \lambda_1$.

We will show that the given problems have infinitely many solutions with all positive components and infinitely many solutions with all negative components. With certain restrictions on c we provide more detailed information about the solution set. In particular, for the $n = 3$ case we prove that the solutions can be characterized as local minima and saddle points of the associated functional.

Our research is motivated by Schaaf and Schmitt (1988) who proved that there exist infinitely many positive, and infinitely many negative, solutions to the boundary value problem,

$$\begin{gathered} -u''(x) = u(x) - c\sin(u(x)), \\ u(0) = 0 = u(\pi). \end{gathered} \tag{1.4}$$

Received by the editors May 4, 2015.
2010 *Mathematics Subject Classification.* 15A24; 35P30; 65H10.
Key words and phrases. Discrete system; nonlinear; resonance.

This problem posed a challenge to nonlinear analysts for several reasons. First, it is a so-called *resonance* problem, i.e. if the nonlinear term $\sin(u(x))$ is replaced by a function $f(x)$, then (1.4) becomes a linear problem which only has solutions if $f(x)$ satisfies the orthogonality condition

$$\int_0^\pi f(x)\sin(x)dx = 0,$$

and in that case has infinitely many solutions. This is a consequence of the Fredholm Alternative, see Drábek and Milota (2013). Second, the term $\sin(u(x))$ does not satisfy the well-known Landesman-Lazer condition, (Landesman and Lazer, 1970), which is a nonlinear adaptation of the previously mentioned orthogonality condition. Our problems are a discrete analog of (1.4).

Schaaf and Schmitt (1988) employed the method of bifurcation from infinity combined with some very careful integral estimates. The bifurcation from infinity argument can also be applied to the discrete problem in a straight-forward way, but the careful integral estimates do not readily adapt to the analogous sums that arise in the discrete problems. Thus it was necessary to find another approach.

We use symmetry and other elementary methods to show that the problems have infinitely many positive and infinitely many negative solutions. For the $n = 3$ case we also reformulate the problem as a search for the critical points of a real valued function $F(x)$, i.e. we employ a variational method. We show that the solutions of the problem can be viewed as minima and saddle points of F. It is this variational approach that has promise of generalizing to higher dimensions. It is interesting to note that, in the absence of the integral estimates mentioned above, a topological theorem related to linear flow on a torus, see Katok and Hasselblatt (1997), provides the structure needed to complete the proof. A few auxiliary results are provided in Section 2, and proofs of the main results are given in Section 3.

2. Auxiliary results

2.1. A topological Theorem

Consider the lattice points in $\mathbb{R}^3$ given by $(2\pi i, 2\pi j, 2\pi k)$ for $i, j, k \in \mathbb{Z}$. We say that two vectors, $v, w \in \mathbb{R}^3$ are equivalent mod 2π, i.e. $v \equiv w[2\pi]$, if $v - w = (2\pi i, 2\pi j, 2\pi k)$ for some $i, j, k \in \mathbb{Z}$. For a given n we can cover $\mathbb{R}^3$ with a collection of cubes

$$C_n^{i,j,k} := \left\{ x \in \mathbb{R}^3 : |x_1 - \frac{2\pi i}{n}| \le \frac{\pi}{n}, |x_2 - \frac{2\pi j}{n}| \le \frac{\pi}{n}, |x_3 - \frac{2\pi k}{n}| \le \frac{\pi}{n} \right\}.$$

We say that $v \in C_n^{i,j,k}[2\pi]$ if there is a $w \in C_n^{i,j,k}$ such that $v \equiv w[2\pi]$. Observe if $v \in \mathbb{R}^3$, then $v \in C_n^{i,j,k}[2\pi]$ for some $i, j, k \in \mathbb{Z}$ such that $0 \le i \le n-1, 0 \le j \le n-1$,and $0 \le k \le n-1$.

Theorem 2.1. *Let $v \in \mathbb{R}^3$ be any nontrivial vector and let $n \in \mathbb{N}$. There exists a sequence $(t_m) \in \mathbb{R}$ such that $t_m \to \infty$ and $t_m v \in C_n^{0,0,0}[2\pi]$ for all m.*

Proof. Consider the cubes $C_{2n}^{i,j,k}$ for $i, j, k \in \mathbb{Z}$. Observe that if $v, w \in C_{2n}^{i,j,k}$, a cube with edgelength $\frac{\pi}{n}$, then $v - w \in C_n^{0,0,0}$. Thus if $v, w \in C_{2n}^{i,j,k}[2\pi]$, then $v - w \in C_n^{0,0,0}[2\pi]$. For every $s \in \mathbb{R}$ there are integers $i, j, k \in \{0, 1, ..., 2n-1\}$ such that $sv \in C_{2n}^{i,j,k}[2\pi]$. Since $\{C_{2n}^{i,j,k}[2\pi] : 0 \le i, j, k \le 2n-1\}$ represents a finite collection of sets, there must be a particular $i, j, k \in \{1, 2, ..., 2n-1\}$ and a sequence (s_m) such that $s_m \to \infty$ and $s_m v \in C_{2n}^{i,j,k}[2\pi]$ for all m. It follows that $(s_m - s_1)v \in C_n^{0,0,0}[2\pi]$ for all m. Let $t_m = s_m - s_1$. □

2.2. Variational methods

The general problem (1.3) has a solution if and only if $F(x) = \frac{1}{2}\langle Ax, x\rangle - \frac{\lambda_1}{2}\langle x, x\rangle - cV(x)$ has a critical point, where V is the potential function with gradient v, and where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. Finding critical points of F is equivalent to finding solutions of the original problem.

We will use the Implicit Function Theorem to reduce our search for critical points to a special curve.

Theorem 2.2 (Implicit Function Theorem,Taylor and Mann (1983); Kaplan (1991))**.** *Let $G_1(x)$ and $G_2(x)$ be continuously differentiable real-valued functions on $\mathbb{R}^3$. Assume that $x^0 \in \mathbb{R}^3$ such that $G_1(x^0) = 0$ and $G_2(x^0) = 0$ and assume that the matrix*

$$\begin{bmatrix} \frac{\partial G_1}{\partial x_2}(x^0) & \frac{\partial G_1}{\partial x_3}(x^0) \\ \frac{\partial G_2}{\partial x_2}(x^0) & \frac{\partial G_2}{\partial x_3}(x^0) \end{bmatrix}$$

is invertible. Then there is a smooth curve $\alpha(t) = (t + x_1^0, x_2(t), x_3(t))$ on an interval $-\epsilon < t < \epsilon$ such that $G_1(\alpha(t)) = G_2(\alpha(t)) = 0$ for all $t \in (-\epsilon, \epsilon)$ and $\alpha(0) = x^0$.

To study the concavity of a function we consider properties of its Hessian matrix.

Theorem 2.3 (Marsden and Hoffman (1993), Section 6.9)**.** *Let $G(x)$ be a continuously differentiable real-valued function on $\mathbb{R}^2$. Assume that the Hessian of G,*

$$H = \begin{bmatrix} \frac{\partial^2 G}{\partial x_1^2} & \frac{\partial^2 G}{\partial x_2 \partial x_1} \\ \frac{\partial^2 G}{\partial x_1 \partial x_2} & \frac{\partial^2 G}{\partial x_2^2} \end{bmatrix},$$

is positive definite, i.e., that there is a $\delta > 0$ such that $\langle Hx, x\rangle \geq \delta\|x\|^2$ for all $x \in \mathbb{R}^2$. Then G is strictly convex and has a unique critical point which is a global minimum.

3. Theoretical results

3.1. The $n = 2$ case

Consider problem (1.1). The given equation is equivalent to the two scalar equations

$$x_1 - x_2 + c\sin(x_1) = 0, \text{ and}$$
$$x_2 - x_1 + c\sin(x_2) = 0.$$

It follows that $x_2 = x_1 + c\sin(x_1)$ and $x_1 = x_2 + c\sin(x_2)$, so we must solve the single equation $\sin(x_1) + \sin(x_1 + c\sin(x_1)) = 0$. This equation has solutions $x_1^k = k\pi$ for all $k \in \mathbb{Z}$. Note that x_1^k and the corresponding $x_2^k = x_1^k$ are either both positive or both negative.

Since this problem is nonlinear it is not, in general, a simple task to determine the complete set of solutions. However, for the special case where $|c| \leq 1$ we can show that the solutions described above are the only solutions. Let $f(t) = t + c\sin(t)$. A solution pair must simultaneously be of the form $(x_1, f(x_1))$ and $(f(x_2), x_2)$. If $|c| \leq 1$, then $f'(t) = 1 + c\cos(t) \geq 1 - |c| \geq 0$ and $f'(t) = 0$ only if $c = \pm 1$ and t is either of the form $2k\pi$ for $c = -1$, or $(2k+1)\pi$ for $c = 1$, where $k \in \mathbb{Z}$. Since $f'(t)$ is nonnegative, and is only 0 at isolated points, f is strictly increasing and invertible. In this case our solutions must satisfy $f(x_1) = f^{-1}(x_1)$. It is clear that, for any $k \in \mathbb{Z}$, we have either $f(x_1) > x_1$ for all x_1 in $(k\pi, (k+1)\pi)$, or $f(x_1) < x_1$ for all x_1 in $(k\pi, (k+1)\pi)$. On the same intervals we have $f^{-1}(x_1) < x_1$ and $f^{-1}(x_1) > x_1$, respectively. Thus the only solution pairs occur where $x_2 = x_1 = k\pi$.

If $|c| > 1$, then the situation gets more complicated, and graphical evidence indicates that the number of solutions increases as $|c|$ increases. See Figure 3.1.

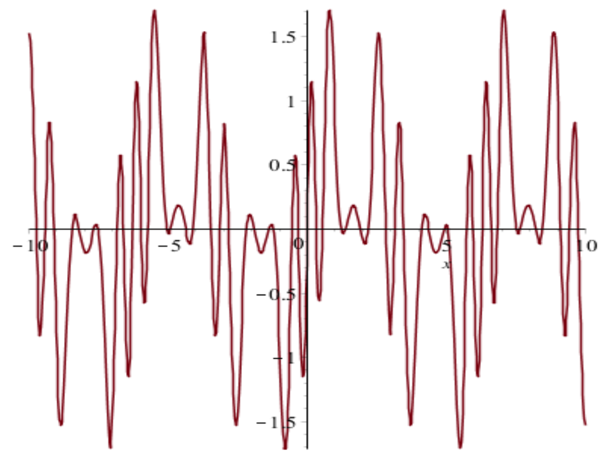

FIGURE 3.1. Graph of $\sin(x_1) + \sin(x_1 + 10\sin(x_1))$, $c = 10$

3.2. The $n = 3$ case

Now we consider problem (1.2). After simplifying the scalar equations, we get

$$\begin{aligned} &x_2 = \sqrt{2}x_1 + c\sin(x_1), \\ &x_2 = \sqrt{2}x_3 + c\sin(x_3), \text{ and} \\ &-x_1 - x_3 + \sqrt{2}x_2 + c\sin(x_2) = 0. \end{aligned}$$

We take advantage of symmetry by setting $x_3 = x_1$ and then substituting for both x_2 and x_3 in the third equation to get

$$\sqrt{2}\sin(x_1) + \sin(\sqrt{2}x_1 + c\sin(x_1)) = 0.$$

Let $h(t) = \sqrt{2}\sin(t) + \sin(\sqrt{2}t + c\sin(t))$. It is clear that $h(\frac{\pi}{2} + 2k\pi) \geq \sqrt{2} - 1 > 0$ for all $k \in \mathbb{Z}$ and $h(\frac{-\pi}{2} + 2k\pi) \leq -\sqrt{2} + 1 < 0$ for all $k \in \mathbb{Z}$. By the Intermediate Value Theorem there must be a sequence, (t_k),with $k \in \mathbb{Z}$, such that $t_k \to \pm\infty$ as $k \to \pm\infty$. The corresponding solutions, $x^k = (t_k, \sqrt{2}t_k + c\sin(t_k), t_k)$, have all positive (negative) components as $k \to \infty$ $(-\infty)$.

We have not argued thus far that the solutions with $x_1 = x_3$ are the only possible solutions. This turns out to be true if we assume a restriction on c.

Lemma 3.1. *If $|c| \leq \sqrt{2}$, then $x_1 = x_3$ is a necessary condition for solving the given equations.*

Proof. Let $f(t) = \sqrt{2}t + c\sin(t)$. So, $f'(t) = \sqrt{2} + c\cos(t) \geq 0$. Moreover, $f'(t) = 0$ only if $c = \pm\sqrt{2}$ and $t = (2k+1)\pi$ or $t = 2k\pi$, respectively, where $k \in \mathbb{Z}$. Thus f is strictly increasing and invertible. Thus the equations $\sqrt{2}x_1 + c\sin(x_1) = x_2 = \sqrt{2}x_3 + c\sin(x_3)$ imply that $x_1 = x_3$. $\square$

Similar to the $n = 2$ case, graphical evidence indicates that there are more solutions when $|c|$ is large.

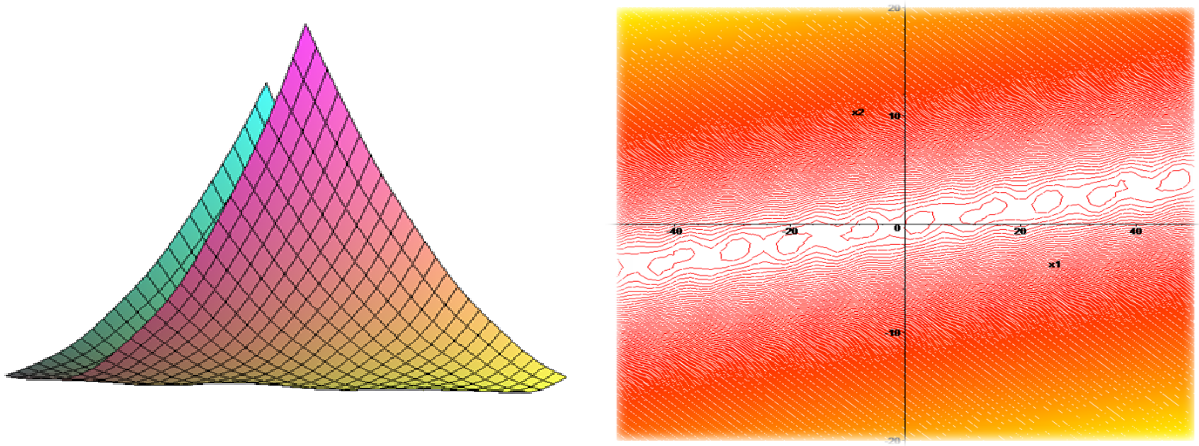

FIGURE 3.2. Two dimensional graph of F and its contour map.

3.3. The $n = 3$ case using variational methods

We present a different analysis of the $n = 3$ case. Although the result we prove is not an improvement in determining the number of solutions, this argument does provide a blue print for possible generalizations to higher dimensions. It also provides a characterization of solutions as minima and saddle points which can be useful when determining further properties of solutions such as the Morse index.

Once again consider problem (1.2) and let

$$F(x) = \frac{1}{2}\langle Ax, x\rangle - \frac{\lambda_1}{2}\langle x, x\rangle - c(\cos(x_1) + \cos(x_2) + \cos(x_3)). \tag{3.1}$$

Let $\{e_1, e_2, e_3\}$ be the set of orthonormal eigenvectors of A that provide a basis for $\mathbb{R}^3$, and let the jth component of e_i be represented as e_{ij}. We write $x = \overline{x}_1 e_1 + \overline{x}_2 e_2 + \overline{x}_3 e_3$. So

$$\begin{aligned} x_1 &= \overline{x}_1 e_{11} + \overline{x}_2 e_{21} + \overline{x}_3 e_{31}, \\ x_2 &= \overline{x}_1 e_{12} + \overline{x}_2 e_{22} + \overline{x}_3 e_{32}, \text{ and} \\ x_3 &= \overline{x}_1 e_{13} + \overline{x}_2 e_{23} + \overline{x}_3 e_{33}. \end{aligned}$$

With our new variables defined, F can be written as

$$F(x) = F(\overline{x}) = \frac{1}{2}\big((\lambda_2 - \lambda_1)\overline{x}_2^2 + (\lambda_3 - \lambda_1)\overline{x}_3^2\big) - c\big(\cos(x_1) + \cos(x_2) + \cos(x_3)\big), \tag{3.2}$$

where x_1, x_2, and x_3 are now functions of $\overline{x}_1, \overline{x}_2$, and $\overline{x}_3$. It is helpful to look at the two parts of F separately. First, it is clear that $\frac{1}{2}((\lambda_2 - \lambda_1)\overline{x}_2^2 + (\lambda_3 - \lambda_1)\overline{x}_3^2)$ is nonnegative, and is strictly convex with respect to $\overline{x}_2$ and $\overline{x}_3$. Second, $-c(\cos(x_1) + \cos(x_2) + \cos(x_3))$ is 2π periodic in each variable and is bounded between $-3c$ and $3c$. Much of what follows will simply verify that the graph of F has the properties seen in the Figure 3.2, where we see a trough-like figure with alternating minima and saddles along the base of the trough.

The theorem that we will prove is

Theorem 3.2. *For small enough c problem* (1.2) *has infinitely many solutions with all positive components and infinitely many solutions with all negative components. Moreover, infinitely many of these solutions can be characterized as local minima of F and infinitely many as saddle points of F.*

The partial derivatives of F are

$$\frac{\partial F}{\partial \overline{x}_1} = c\big(\sin(x_1)e_{11} + \sin(x_2)e_{12} + \sin(x_3)e_{13}\big),$$
$$\frac{\partial F}{\partial \overline{x}_2} = (\lambda_2 - \lambda_1)\overline{x}_2 + c\big(\sin(x_1)e_{21} + \sin(x_2)e_{22} + \sin(x_3)e_{23}\big), \text{ and}$$
$$\frac{\partial F}{\partial \overline{x}_3} = (\lambda_3 - \lambda_1)\overline{x}_3 + c\big(\sin(x_1)e_{31} + \sin(x_2)e_{32} + \sin(x_3)e_{33}\big).$$

Consider the equations $\frac{\partial F}{\partial \overline{x}_2} = 0$ and $\frac{\partial F}{\partial \overline{x}_3} = 0$. Observe that

$$\frac{\partial^2 F}{\partial \overline{x}_2^2} = (\lambda_2 - \lambda_1) + c\big(\cos(x_1){e_{21}}^2 + \cos(x_2){e_{22}}^2 + \cos(x_3){e_{23}}^2\big),$$
$$\frac{\partial^2 F}{\partial \overline{x}_3 \partial \overline{x}_2} = c\big(\cos(x_1)e_{21}e_{31} + \cos(x_2)e_{22}e_{32} + \cos(x_3)e_{23}e_{33}\big),$$
$$\frac{\partial^2 F}{\partial \overline{x}_3 \partial \overline{x}_2} = \frac{\partial^2 F}{\partial \overline{x}_2 \partial \overline{x}_3}, \text{ and}$$
$$\frac{\partial^2 F}{\partial \overline{x}_3^2} = (\lambda_3 - \lambda_1) + c\big(\cos(x_1){e_{31}}^2 + \cos(x_2){e_{32}}^2 + \cos(x_3){e_{33}}^2\big).$$

Observe that

$$|\cos(x_1){e_{21}}^2 + \cos(x_2){e_{22}}^2 + \cos(x_3){e_{23}}^2| \le {e_{21}}^2 + {e_{22}}^2 + {e_{23}}^2 = 1.$$

Similarly,

$$|\cos(x_1){e_{31}}^2 + \cos(x_2){e_{32}}^2 + \cos(x_3){e_{33}}^2| \le 1,$$

and

$$\begin{aligned} |\cos(x_1)e_{21}e_{31} + \cos(x_2)e_{22}e_{32} + \cos(x_3)e_{23}e_{33}| &\le |e_{21}e_{31}| + |e_{22}e_{32}| + |e_{23}e_{33}| \\ &\le (e_{21}^2 + e_{22}^2 + e_{23}^2)^{\frac{1}{2}}(e_{31}^2 + e_{32}^2 + e_{33}^2)^{\frac{1}{2}} \\ &\le 1, \end{aligned}$$

where we have applied the Cauchy-Schwartz inequality. Thus we can write

$$M = \begin{bmatrix} \frac{\partial^2 F}{\partial \overline{x}_2^2} & \frac{\partial^2 F}{\partial \overline{x}_3 \partial \overline{x}_2} \\ \frac{\partial^2 F}{\partial \overline{x}_2 \partial \overline{x}_3} & \frac{\partial^2 F}{\partial \overline{x}_3^2} \end{bmatrix} = \begin{bmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & \lambda_3 - \lambda_1 \end{bmatrix} + c \begin{bmatrix} C_1 & C_2 \\ C_2 & C_3 \end{bmatrix}$$

where $|C_i| \le 1$ for $i = 1, 2, 3$. The first matrix on the right is clearly a diagonal matrix that is positive definite and invertible. For small enough c, it is clear by continuity that M is positive definite and invertible. It follows that F is strictly convex in the variables $\overline{x}_2$ and $\overline{x}_3$. Thus for each fixed $\overline{x}_1$ there is a unique point $(\overline{x}_1, \overline{x}_2(\overline{x}_1), \overline{x}_3(\overline{x}_1))$, where

$$\frac{\partial F}{\partial \overline{x}_2} = \frac{\partial F}{\partial \overline{x}_3} = 0,$$

and this point locates a global minimum. Hence we have found a curve $\alpha(t) = (t, \overline{x}_2(t), \overline{x}_3(t))$ for $t \in \mathbb{R}$, such that all critical points of F must be contained on the curve. Since M is invertible, we can apply the Implicit Function Theorem to conclude that $\alpha(t)$ is differentiable.

Observe that for fixed $\overline{x}_1$ we have that

$$F(\overline{x}_1, 0, 0) = -c(\cos(x_1) + \cos(x_2) + \cos(x_3)) \leq 3|c|.$$

If either $|\overline{x}_2| \geq \sqrt{\frac{14|c|}{\lambda_2-\lambda_1}}$ or $|\overline{x}_3| \geq \sqrt{\frac{14|c|}{\lambda_3-\lambda_1}}$, then substituting into (3.2) gives $F(\overline{x}) \geq 4|c| > F(\overline{x}_1, 0, 0)$, and so the global minimum associated with $\overline{x}_1$ must satisfy

$$|\overline{x}_2| \leq \sqrt{\frac{14|c|}{\lambda_2 - \lambda_1}} \text{ and } |\overline{x}_3| \leq \sqrt{\frac{14|c|}{\lambda_3 - \lambda_1}}.$$

Thus the $\overline{x}_2$ and $\overline{x}_3$ components of $\alpha(t)$ remain bounded as $t \to \pm\infty$.

Our next goal is to show that F restricted to the curve $\alpha(t)$ has infinitely many minima and maxima as $t \to \infty$. The minima of F restricted to α will be local minima of F on $\mathbb{R}^3$, and the maxima of F restricted to α will be saddle points of F on $\mathbb{R}^3$. We note that by construction $\frac{\partial F}{\partial \overline{x}_2} = 0$ and $\frac{\partial F}{\partial \overline{x}_3} = 0$ at each point on $\alpha(t)$. At each restricted minima or maxima we will also have

$$0 = \frac{dF}{dt} = \left\langle \nabla F(\alpha(t)), \alpha'(t) \right\rangle = \left\langle \left(\frac{\partial F}{\partial \overline{x}_1}(\alpha(t)), 0, 0 \right), \left(1, \overline{x}_2'(t), \overline{x}_3'(t)\right) \right\rangle = \frac{\partial F}{\partial \overline{x}_1}(\alpha(t)),$$

so we must have $\frac{\partial F}{\partial \overline{x}_1} = 0$. This verifies that critical points of F restricted to $\alpha(t)$ will also be critical points of F on $\mathbb{R}^3$.

It remains to verify that F restricted to $\alpha(t)$ must rise and fall infinitely many times as $t \to \pm\infty$. We assume $c > 0$, and note that the opposite case can be handled similarly. It is useful to first examine F restricted to the line te_1. Using the eigenvalue and eigenvector properties it is clear that $A(te_1) - \lambda_1(te_1) = 0$. It follows that

$$F(te_1) = c\big(-\cos(te_{11}) - \cos(te_{12}) - \cos(te_{31})\big),$$

which is $-3c$ at $t = 0$ and is bounded between $-3c$ and $3c$ for other t. By the minimizing property of $\alpha(t)$ we have $F(\alpha(t)) \leq F(te_1)$.

Now we apply Theorem 2.1. Since $F(0, 0, 0) = -3c$ we can apply continuity to choose n large enough so that $F(x) \leq -2c$ in the cube $C_n^{0,0,0}$. For each $t \in \mathbb{R}$ such that $te_1 \in C_n^{0,0,0}[2\pi]$ we know that $F(\alpha(t)) \leq F(te_1) \leq -2c$. Hence $F(\alpha(t)) \leq -2c$ for infinitely many t as $t \to \pm\infty$.

Consider $x_1(t) = te_{11} + \overline{x}_2(t)e_{21} + \overline{x}_3(t)e_{31}$. We know that $\overline{x}_2(t)$ and $\overline{x}_3(t)$ are bounded and e_{11} is positive, so $x_1(t) \to \pm\infty$ as $t \to \pm\infty$. It follows that the term $\cos(x_1(t))$ will oscillate between -1 and 1 infinitely many times. When $\cos(x_1(t)) = -1$ we have that

$$-c\big(\cos(x_1(t)) + \cos(x_2(t)) + \cos(x_3(t))\big) \geq -c$$

and so $F(\alpha(t)) \geq -c$.

Thus we have shown that $F(\alpha(t)) \leq -2c$ infinitely many times and $F(\alpha(t)) \geq -c$ infinitely many times as $t \to \infty$. It follows that $F(\alpha(t))$ achieves infinitely many minima and maxima. Moreover, our solutions are of the form $te_1 + \overline{x}_2(t)e_2 + \overline{x}_3(t)e_3$. Since $\overline{x}_2(t)$ and $\overline{x}_3(t)$ are bounded, and e_1 has all positive components, it is clear that the solutions have either all positive or all negative components for large t.

4. Conclusion

We have proved that results analogous to those of Schaaf and Schmitt are true for the discrete problems (1.1) and (1.2). Moreover, we were able to characterize solutions as saddle points and local minima.

A number of questions remain open. Can we remove the restrictions on c? Can we generalize A to be any symmetric and positive definite matrix? Do the arguments generalize to higher dimensions? It is clear that further investigation of the connection between the discrete and continuous problems is warranted and might provide greater insight into the ODE problem that motivated this paper in the first place.

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