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A Mixed Finite Element Approximation of pre-Darcy Flows

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ABSTRACT. In this paper, we consider the pre-Darcy flows for slightly compressible fluids. Using Muskat's and Ward's general form of Forchheimer equations, we describe the fluid dynamics by a nonlinear system of density and momentum. A mixed finite element method is proposed for the approximation of the solution of the above system. The stability of the approximations are proved; the error estimates are derived for the numerical approximations for both continuous and discrete time procedures. Numerical experiments confirm the theoretical analysis regarding convergence rates.

1. Introduction

Fluid flows through porous media are encountered in a wide range of science and engineering applications, e.g., water resources, geothermal systems, chemical processes, gas and water purification, gas storage, oil extraction, and petroleum engineering, chemical engineering, mining and mineral processing, and oil and gas production. Due to the variety and complexity of the filtration matrix, they can be very complicated and can be modeled, depending on each situation, by a number of equations of various types with different parameters. Broadly speaking, they are categorized into three known regimes, namely, pre-Darcy flow (i.e, pre-linear, non-Darcy), Darcy flow (linear) and post-Darcy flow (i.e, post-linear, non-Darcy). There is a general consensus that the Darcy regime is valid as long as the Reynolds number (Re) is in the range of characteristic values between 1 and 10 (see Bear (1972)). When the Reynolds number is high ($Re > 10$), there is a deviation from Darcy's law, and the Forchheimer equations are usually used to account for it (see Forchheimer (1901); Muskat (1937); Bear (1972); Nield and Bejan (2013)). At the other end of the Reynolds number range, when it is very small ($Re \to 0$) (see Boettcher et al. (2022); Farmani et al. (2018)) the pre-Darcy regime is observed but not fully understood, although it contributes to the unexpected extraction of crude oil and improved recovery in petroleum reservoirs (see Dudgeon (1985); Siddiqui et al. (2016); Soni et al. (1978); Bloshanskaya et al. (2017) and references therein).

We now start to investigate the pre-Darcy fluid flows in porous media. Consider fluid flows with velocity $\mathbf{v} \in \mathbb{R}^d$, $d \geq 2$ pressure $p \in \mathbb{R}$, and density $\rho \in [0, \infty)$. As the flow rate (Reynolds's number) is sufficiently small, Izbash (see Izbash (1931)) presented an equation describing the pre-Darcy regime of the form:

$$|\mathbf{v}(\mathbf{x}, t)|^{-\alpha}\mathbf{v}(\mathbf{x}, t) = -k(\mathbf{x}, t)\nabla p(\mathbf{x}, t) \tag{1.1}$$

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for some constant power $\alpha \in (0,1)$ and coefficient $k(\mathbf{x},t) > 0$. For experimental values of α (see e.g. Siddiqui et al. (2016); Soni et al. (1978)).

In order to take into account the presence of density in the generalized Forchheimer equation, we modify (1.1) using dimension analysis by Muskat Muskat (1937) and Ward Ward (1964). They proposed the following equation for both laminar and turbulent flows in porous media:

$$-\nabla p(\mathbf{x},t) = G\big(\mathbf{v}^i \kappa^{\frac{i-3}{2}} \rho^{i-1} \mu^{2-i}\big), \tag{1.2}$$

where G is a function of one variable, $\mu = \mu(\mathbf{x},t)$ is the viscosity of the fluid, $\kappa = \kappa(\mathbf{x},t)$ is the permeability of the medium.

In particular, when $i = 1$, Ward Ward (1964) established the Darcy's law to match the experimental data

$$-\nabla p(\mathbf{x},t) = \frac{\mu(\mathbf{x},t)}{\kappa(\mathbf{x},t)} \mathbf{v}(\mathbf{x},t), \tag{1.3}$$

and when $i = 2$ for Forchheimer's law

$$-\nabla p(\mathbf{x},t) = \frac{\mu(\mathbf{x},t)}{\kappa(\mathbf{x},t)} \mathbf{v}(\mathbf{x},t) + c_F \frac{\rho(\mathbf{x},t)}{\sqrt{\kappa(\mathbf{x},t)}} |\mathbf{v}(\mathbf{x},t)| \mathbf{v}(\mathbf{x},t), \quad \text{where } c_F > 0. \tag{1.4}$$

Combining (1.1) with the suggestive form (1.2) for the dependence on ρ and $\mathbf{v}$, we propose the following equation

$$-\nabla p(\mathbf{x},t) = a(\mathbf{x},t)\rho(\mathbf{x},t)^{-\alpha} |\mathbf{v}(\mathbf{x},t)|^{-\alpha} \mathbf{v}(\mathbf{x},t), \tag{1.5}$$

where $a(\mathbf{x},t)$ is a positive function.

Multiplying both sides of the equation (1.5) by ρ, we find that

$$\big(a(\mathbf{x},t)|\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)|^{-\alpha}\big)\, \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t) = -\rho(\mathbf{x},t)\nabla p(\mathbf{x},t). \tag{1.6}$$

Under isothermal conditions, the state equation only relates the density ρ with the pressure p, that is, $\rho = \rho(p)$. Therefore, the equation of state for slightly compressible fluids is given by

$$\frac{d\rho}{dp} = \frac{\rho}{\bar{\omega}},$$

where $1/\bar{\omega} > 0$ represents the small compressibility.

Hence,

$$\nabla \rho = \frac{1}{\bar{\omega}} \rho \nabla p, \quad \text{or} \quad \rho \nabla p = \bar{\omega} \nabla \rho. \tag{1.7}$$

Combining (1.6) and (1.7) implies that

$$a(\mathbf{x},t)|\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)|^{-\alpha} \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t) = -\bar{\omega}\nabla\rho(\mathbf{x},t). \tag{1.8}$$

The continuity equation is

$$\phi \rho_t(\mathbf{x},t) + \mathrm{div}(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)) = f(\mathbf{x},t), \tag{1.9}$$

where $\phi \in (0,1)$ is the constant porosity and f is external mass flow rate.
By combining (1.8) and (1.9), we have

$$a(\mathbf{x},t)|\mathbf{m}(\mathbf{x},t)|^{-\alpha}\mathbf{m}(\mathbf{x},t) = -\bar{\omega}\nabla\rho(\mathbf{x},t),$$
$$\phi\rho_t(\mathbf{x},t) + \mathrm{div}\ \mathbf{m}(\mathbf{x},t) = f(\mathbf{x},t),$$

where $\mathbf{m}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)$.

By rescaling the variable $\rho(\mathbf{x},t) \to \phi\rho(\mathbf{x},t)$, $a(\mathbf{x},t) \to \bar{\omega}^{-1}\phi a(\mathbf{x},t)$, we obtain a system of equations

$$\begin{aligned} a(\mathbf{x},t)|\mathbf{m}(\mathbf{x},t)|^{-\alpha}\mathbf{m}(\mathbf{x},t) &= -\nabla\rho(\mathbf{x},t), \\ \rho_t(\mathbf{x},t) + \text{div }\mathbf{m}(\mathbf{x},t) &= f(\mathbf{x},t). \end{aligned} \tag{1.10}$$

We are interested in solving equation (1.10) endowed with initial and boundary conditions

$$\begin{aligned} a(\mathbf{x},t)|\mathbf{m}(\mathbf{x},t)|^{-\alpha}\mathbf{m}(\mathbf{x},t) &= -\nabla\rho(\mathbf{x},t) && (\mathbf{x},t) \in \Omega \times I, \\ \rho_t(\mathbf{x},t) + \nabla\cdot\mathbf{m}(\mathbf{x},t) &= f(\mathbf{x},t) && (\mathbf{x},t) \in \Omega \times I, \\ \rho(\mathbf{x},t) &= 0 && (\mathbf{x},t) \in \partial\Omega \times I, \\ \rho(\mathbf{x},0) &= \rho_0(\mathbf{x}) && \mathbf{x} \in \Omega. \end{aligned} \tag{1.11}$$

Throughout the paper, we make the following assumptions on the data and coefficients:

(H1) The coefficients $a \in W^{1,\infty}(I, L^\infty(\Omega))$ satisfy $0 < a_* < a(\mathbf{x},t) < a^* < \infty$, $|a_t(\mathbf{x},t)| < b^* < \infty$ for almost every $(\mathbf{x},t) \in \bar{\Omega} \times I$.
(H2) $f \in W^{1,\infty}(I, L^2(\Omega))$.

It is well known that the mixed finite element methods and related cell-centered finite difference methods have become popular in recently years for modeling flow in porous media because they produce the accurate results for both scalar (density) and vector (momentum) functions (see Park (2005)). An analysis of mixed finite element method to a Darcy-Forchheimer steady state model was well studied in Pan and Rui (2012); Salas et al. (2013). The mixed methods for a nondegenerate system modeling flows in porous media were studied in Dawson and Wheeler (1994); Park (2005); Girault and Wheeler (2008); Kim and Park (1999). The authors in Arbogast et al. (1996); Woodward and Dawson (2000); Fadimba and Sharpley (1995, 2004) analyzed the mixed finite element approximations of the non-linear degenerate system modeling the water-gas flow in porous media. In their analysis, the Kirchhoff transformation is used to move the nonlinearity from coefficients to the gradient.

In this paper, we analyze mixed finite element approximations to the solutions of the system of equations modeling the flows of a single-phase compressible fluid in porous media subject to the pre-Darcy law. We mention Cummings et al. (2024); Girault and Wheeler (2008); Kieu (2018); Park (2005) for a mixed finite element discretization of (1.11). This is a nonlinear system with coefficients depending on the momentum. The Kirchhoff transformation is not applicable to this system. For our equations, we combine the techniques developed in our previous work in Hoang and Kieu (2017, 2019); Hoang et al. (2014); Ibragimov and Kieu (2016); Kieu (2015, 2020) and utilize the special structures of the equations to obtain the stability of the approximate solution. The error estimates are derived for the numerical approximations of the density and momentum in both continuous and discrete time procedures.

The paper is organized as follows. Section 2 introduces the notations and the relevant results. Section 3 establishes many estimates of the energy type norms for the solution $(\mathbf{m},\rho)$ to the initial boundary value problem (IBVP) (3.1) in Lebesgue norms, expressed in terms of the boundary data and the initial data. Section 4 presents a semidiscrete mixed finite element approximation for the IBVP (1.11). We discuss the existence and uniqueness and derive error estimates. The fully discrete scheme is considered in section 5, where the error estimates are derived in terms of the discretization parameters. In section 6, the results of a few numerical experiments using the lowest Raviart-Thomas mixed finite element in the two dimensions are reported. These results support our theoretical analysis regarding convergence rates.

2. Notations and preliminary results

Throughout this paper, we assume that Ω is an open bounded subset of $\mathbb{R}^d$, with $d = 2, 3, \ldots$, and has C^0-boundary $\partial\Omega$. For $s \in [0, \infty)$, we denote by $L^s(\Omega)$ the set of s-integrable functions on Ω and $(L^s(\Omega))^d$ the space of d-dimensional vectors which have all components in $L^s(\Omega)$. We denote $\langle \cdot, \cdot \rangle$ the inner product in either $L^s(\Omega)$ or $(L^s(\Omega))^d$ that is $\langle \xi, \eta \rangle = \int_\Omega \xi \eta dx$ or $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \int_\Omega \boldsymbol{\xi} \cdot \boldsymbol{\eta} dx$ and $\|v\|_{0,s} = \left(\int_\Omega |v(x)|^s dx \right)^{1/s}$ for standard Lebesgue norm of the measurable function. For $m \geq 0, s \in [0, \infty]$, we denote the Sobolev spaces by $W^{m,s}(\Omega) = \{v \in L^s(\Omega) : D^\alpha v \in L^s(\Omega), |\alpha| \leq m\}$ and the norm of $W^{m,s}(\Omega)$ by $\|v\|_{m,s} = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha v|^s dx \right)^{1/s}$, and $\|v\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \operatorname{ess\,sup}_\Omega |D^\alpha v|$. Let $I = [0, T]$, we define $L^s(I, X)$ to be the space of all measurable functions $v : I \to X$ with the norm $\|v\|_{L^s(I,X)} = \left(\int_0^T \|v(t)\|_X^s \, dt \right)^{1/s}$, and $L^\infty(I; X)$ to be the space of all measurable functions $v : I \to X$ such that $v : t \to \|v(t)\|_X$ is essentially bounded on I with the norm $\|v\|_{L^\infty(I,X)} = \operatorname{ess\,sup}_{t \in I} \|v(t)\|_X$. We use short hand notations, $\|\rho(t)\| = \|\rho(\cdot, t)\|_{0,2}, \forall t \geq 0$ and $\rho_0(\cdot) = \rho(\cdot, 0)$.

Our calculations frequently use the following exponents:

$$s = 2 - \alpha, \quad s^* = \frac{s}{s-1}.$$

The argument C will represent positive generic constants and their values depend on exponents, the spatial dimension d and domain Ω, independent of the initial and boundary data and time step. These constants may be different place by place.

We recall below some more elementary inequalities that will be used in this paper. First, for $z \in \mathbb{R}$, denote $z^+ = \max\{0, z\}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $p > 0$, one has

$$\frac{|\mathbf{x}|^p + |\mathbf{y}|^p}{2} \leq (|\mathbf{x}| + |\mathbf{y}|)^p \leq 2^{(p-1)^+}(|\mathbf{x}|^p + |\mathbf{y}|^p). \tag{2.1}$$

By the triangle inequality and the second inequality of (2.1), we have

$$|\mathbf{x} - \mathbf{y}|^p \geq 2^{-(p-1)^+}|\mathbf{x}|^p - |\mathbf{y}|^p \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \ p > 0. \tag{2.2}$$

For any $r \geq 1$, $x_1, x_2, \ldots, x_k \geq 0$,

$$x_1^r + x_2^r + \cdots + x_k^r \leq (x_1 + x_2 + \cdots + x_k)^r \leq k^{r-1}(x_1^r + x_2^r + \cdots + x_k^r). \tag{2.3}$$

The following are some commonly used consequences of Young's inequality. If $x, y \geq 0$, $\gamma \geq \beta \geq \alpha > 0$, $p, q > 1$ with $1/p + 1/q = 1$, and $\varepsilon > 0$, then

$$x^\alpha \leq 1 + x^\beta, \quad x^\beta \leq x^\alpha + x^\gamma, \quad xy \leq \varepsilon x^p + \varepsilon^{-q/p} y^q. \tag{2.4}$$

Above and throughout the paper, we conveniently use $0^0 = 1$.

Lemma 2.1. *Assume $-1 < p \leq 0$, then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,*

$$||\mathbf{x}|^p\mathbf{x} - |\mathbf{y}|^p\mathbf{y}| \leq 2|\mathbf{x} - \mathbf{y}|^{1+p}, \tag{2.5}$$

$$(|\mathbf{x}|^p\mathbf{x} - |\mathbf{y}|^p\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \geq (1 + p)(|\mathbf{x}| + |\mathbf{y}|)^p |\mathbf{x} - \mathbf{y}|^2. \tag{2.6}$$

It is meant, naturally, in (2.5) and (2.6) that

$$|\mathbf{x}|^p\mathbf{x}, |\mathbf{y}|^p\mathbf{y}, (|\mathbf{x}| + |\mathbf{y}|)^p|\mathbf{x} - \mathbf{y}|^2 = 0 \text{ for } p \in (-1, 0) \text{ and } \mathbf{x} = \mathbf{y} = 0.$$

Proof. *Proof of inequality* (2.5) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. The inequality obviously holds true when $\mathbf{x} = 0$ or $\mathbf{y} = 0$ or $\mathbf{x} = \mathbf{y}$. We consider only $\mathbf{x}, \mathbf{y} \neq 0$ and $\mathbf{x} \neq \mathbf{y}$.

In Scenario 1, we can assume $\mathbf{y} = -k\mathbf{x}$ for some $k \geq 0$. We have

$$(|\mathbf{x}|^p\mathbf{x} - |\mathbf{y}|^p\mathbf{y}) = |\mathbf{x}|^{1+p}(1 + k^{1+p}).$$

Since $0 < 1 + p \leq 1$, we have from (2.1) that $1 + k^{1+p} \leq 2(1 + k)^{1+p}$. Hence,

$$(|\mathbf{x}|^p\mathbf{x} - |\mathbf{y}|^p\mathbf{y}) \leq 2|\mathbf{x}|^{1+p}(1 + k)^{1+p} = |\mathbf{x} - \mathbf{y}|^{1+p},$$

which proves (2.5).

In Scenario 2, let $\gamma(\tau) = \tau\mathbf{x} + (1 - \tau)\mathbf{y}, \tau \in [0, 1]$ and $h(\tau) = |\gamma(\tau)|^p\gamma(\tau)$. Then,

$$\begin{aligned} ||\mathbf{x}|^p\mathbf{x} - |\mathbf{y}|^p\mathbf{y}| &= \Big| \int_0^1 h'(\tau)d\tau \Big| \\ &= \Big| \int_0^1 \Big(|\gamma(\tau)|^p(\mathbf{x} - \mathbf{y}) + p|\gamma(\tau)|^{p-1}\frac{\gamma(\tau) \cdot (\mathbf{x} - \mathbf{y})}{|\gamma(\tau)|}\gamma(\tau) \Big) d\tau \Big| \qquad (2.7) \\ &\leq (1 + p)|\mathbf{x} - \mathbf{y}| \int_0^1 |\gamma(\tau)|^p d\tau. \end{aligned}$$

We claim

$$\int_0^1 |\gamma(\tau)|^p d\tau \leq \frac{2}{1 + p}|\mathbf{x} - \mathbf{y}|^p. \qquad (2.8)$$

The inequality (2.5) follows by substituting (2.8) into (2.7).

Proof of claim (2.8)

Consider $|\mathbf{x}| \geq |\mathbf{x} - \mathbf{y}|$, then

$$||\mathbf{x}| - (1 - \tau)|\mathbf{x} - \mathbf{y}|| \leq |\mathbf{x} - (1 - \tau)(\mathbf{x} - \mathbf{y})| = |\tau\mathbf{x} + (1 - \tau)\mathbf{y}|$$

and note that $p < 0$,

$$|\tau\mathbf{x} + (1 - \tau)\mathbf{y}|^p \leq ||\mathbf{x}| - (1 - \tau)|\mathbf{x} - \mathbf{y}||^p \leq ||\mathbf{x} - \mathbf{y}| - (1 - \tau)|\mathbf{x} - \mathbf{y}||^p = \tau^p|\mathbf{x} - \mathbf{y}|^p.$$

This shows that

$$\int_0^1 |\gamma(\tau)^p d\tau \leq |\mathbf{x} - \mathbf{y}|^p \int_0^1 \tau^p d\tau \leq \frac{2}{1 + p}|\mathbf{x} - \mathbf{y}|^p.$$

Consider $|\mathbf{x}| < |\mathbf{x} - \mathbf{y}|$. Let $\tau_* \in (0, 1)$ be defined by $(1 - \tau_*)|\mathbf{x} - \mathbf{y}| = |\mathbf{x}|$

$$\begin{aligned} \int_0^1 |\gamma(\tau)|^p d\tau \leq \int_0^1 ||\mathbf{x}| - (1 - \tau)|\mathbf{x} - \mathbf{y}||^p d\tau &= |\mathbf{x} - \mathbf{y}|^p \int_0^1 |\tau - \tau_*|^p d\tau \\ &= \frac{1}{1 + p}|\mathbf{x} - \mathbf{y}|^p(\tau_*^{1+p} + (1 - \tau_*)^{1+p}) \leq \frac{2}{1 + p}|\mathbf{x} - \mathbf{y}|^p. \end{aligned}$$

Proof of inequality (2.6). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Consider Scenario 1 and define the function

$$\ell(t) = |\gamma(t)|^p\gamma(t) \cdot (\mathbf{x} - \mathbf{y}) \text{ for } t \in [0, 1].$$

Then,

$$\begin{aligned}(|\mathbf{x}|^p\mathbf{x}-|\mathbf{y}|^p\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) &= \int_0^1 \ell'(t)dt \\ &= \int_0^1 |\gamma(t)|^p|\mathbf{x}-\mathbf{y}|^2 + p|\gamma(t)|^{p-2}|\gamma(t)\cdot(\mathbf{x}-\mathbf{y})|^2 dt \\ &\geq (1+p)|\mathbf{x}-\mathbf{y}|^2\int_0^1 |\gamma(t)|^p dt.\end{aligned}$$

Note that $-p\in(0,1]$, and hence $|\gamma(t)|^{-p}\leq(|\mathbf{x}|+|\mathbf{y}|)^{-p}$. Therefore, we obtain (2.6).

In Scenario 2, we can assume $\mathbf{y}=-k\mathbf{x}$ for some $k\geq 0$. We have

$$(|\mathbf{x}|^p\mathbf{x}-|\mathbf{y}|^p\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = |\mathbf{x}|^{2+p}(1+k^{1+p})(1+k).$$

Since $0<1+p<1$, we have from (2.1) that $1+k^{1+p}\geq(1+k)^{1+p}$. Hence,

$$(|\mathbf{x}|^p\mathbf{x}-|\mathbf{y}|^p\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \geq |\mathbf{x}|^{2+p}(1+k)^{2+p} = |\mathbf{x}-\mathbf{y}|^2(|\mathbf{x}|+|\mathbf{y}|)^p,$$

which proves (2.6) again. □

We recall a discrete version of the Grönwall Lemma in backward difference form, which will be useful later.

Lemma 2.2. *Assume the nonnegative sequences* $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$,$\{g_n\}_{n=0}^\infty$ *satisfying*

$$\frac{a_n-a_{n-1}}{\tau}-a_n+b_n\leq g_n, \quad n=1,2,3\dots$$

then for a sufficiently small τ*,*

$$a_n+\tau\sum_{i=1}^n b_i \leq e^{\frac{n\tau}{1-\tau}}\Big(a_0+\tau\sum_{i=1}^n g_i\Big). \tag{2.9}$$

Proof. Let $\bar{a}_n=(1-\tau)^n a_n$. A simple calculation shows that

$$\frac{\bar{a}_n-\bar{a}_{n-1}}{\tau} = (1-\tau)^{n-1}\Big(\frac{a_n-a_{n-1}}{\tau}-a_n\Big)\leq(1-\tau)^{n-1}(g_n-b_n).$$

Summation over n leads to

$$\frac{\bar{a}_n-\bar{a}_0}{\tau}\leq\sum_{i=1}^n(1-\tau)^{i-1}g_i-\sum_{i=1}^n(1-\tau)^{i-1}b_i,$$

which gives

$$\bar{a}_n+\tau\sum_{i=1}^n(1-\tau)^{i-1}b_i\leq\bar{a}_0+\tau\sum_{i=1}^n(1-\tau)^{i-1}g_i,$$

and therefore,

$$a_n+\tau\sum_{i=1}^n(1-\tau)^{i-1-n}b_i\leq(1-\tau)^{-n}\Big(a_0+\tau\sum_{i=1}^n(1-\tau)^{i-1}g_i\Big).$$

Since $(1-\tau)^{i-1-n}=\frac{1}{(1-\tau)^{n-i+1}}>1$ and $(1-\tau)^{-n}\leq e^{\frac{n\tau}{1-\tau}}$.

Therefore, (2.9) holds true. □

3. The mixed finite element approximation

In order to derive the mixed formulation of the problem (1.11), we define the following spaces:

$$Q = L^2(\Omega), \quad \text{and} \quad V = \Big\{ \mathbf{v} \in (L^s(\Omega))^d, \nabla \cdot \mathbf{v} \in L^2(\Omega) \Big\}.$$

The norm of V is defined by $\|\mathbf{v}\|_V = \|\mathbf{v}\|_{0,s} + \|\nabla \cdot \mathbf{v}\|$.

Setting

$$A(\mathbf{v}) = a|\mathbf{v}|^{-\alpha}\mathbf{v}.$$

The mixed formulation of (1.11) reads as follows: Find $(\mathbf{m}, \rho) : I \to V \times Q$ such that

$$\begin{aligned} \langle A(\mathbf{m}), \mathbf{v} \rangle - \langle \rho, \nabla \cdot \mathbf{v} \rangle &= 0 \quad \text{for all } \mathbf{v} \in V, \\ \langle \rho_t, q \rangle + \langle \nabla \cdot \mathbf{m}, q \rangle &= \langle f, q \rangle \quad \text{for all } q \in Q. \end{aligned} \tag{3.1}$$

The existence and uniqueness of weak solutions of (3.1) can be treated by the theory of nonlinear monotone operators, see in Lions (1969); Showalter (1997); Zeidler (1990); Knabner and Summ (2017), and also see in Kieu (2015, 2020) for our proof in the case of the Dirichlet boundary condition. The regularity of weak solutions is treated in Ladyženskaja et al. (1968); Ivanov (1982). For the *priori* estimates, we consider weak solutions with enough regularities in both $\mathbf{x}$ and t variables, but not necessarily classical, so that our calculations can be applied.

Lemma 3.1. *Let* $(\mathbf{m}, \rho)$ *be a solution to the problem* (3.1). *There exists a positive constant* C *such that*

$$\|\rho\|^2_{L^\infty(I,L^2(\Omega))} + \|\mathbf{m}\|^s_{L^s(I,L^s(\Omega))} \le C\mathcal{A}, \tag{3.2}$$

$$\textit{where} \quad \mathcal{A} = \|\rho_0\|^2 + \|f\|^2_{L^\infty(I,L^2(\Omega))}. \tag{3.3}$$

Proof. Choosing $(\mathbf{v}, q) = (\mathbf{m}, \rho)$ in (3.1) and adding the resultant equations yield

$$\frac{1}{2}\frac{d}{dt}\|\rho\|^2 + \langle A(\mathbf{m}), \mathbf{m} \rangle = \langle f, \rho \rangle. \tag{3.4}$$

By (2.6), the second term of (3.4) is bounded from below by

$$\langle A(\mathbf{m}), \mathbf{m} \rangle = \big\langle a|\mathbf{m}|^{-\alpha}\mathbf{m}, \mathbf{m} \big\rangle \ge a_* \|\mathbf{m}\|^s_{0,s}. \tag{3.5}$$

We bound the right hand side of (3.4) by using Young's inequality to obtain

$$\langle f, \rho \rangle \le \frac{1}{2}\|f\|^2 + \frac{1}{2}\|\rho\|^2. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) yield

$$\frac{d}{dt}\|\rho\|^2 + 2a_*\|\mathbf{m}\|^s_{0,s} \le \|\rho\|^2 + \|f\|^2.$$

By Grönwall's inequality, we find that

$$\|\rho\|^2_{L^\infty(I,L^2)} + \|\mathbf{m}_h\|^s_{L^s(I,L^s)} \le \|\rho(0)\|^2 + C\|f\|^2_{L^\infty(I,L^2)}.$$

Thus, the inequality (3.2) holds. □

Lemma 3.2. *Let* $(\mathbf{m}, \rho)$ *be a solution to the problem* (3.1). *Then, there exists a positive constant* C *such that*

$$\|\rho_t\|^2_{L^\infty(I,L^2(\Omega))} + \|\mathbf{m}\|^s_{L^\infty(I,L^s(\Omega))} + \|\nabla \cdot \mathbf{m}\|^2_{L^\infty(I,L^2(\Omega))} \le C\mathcal{B}, \tag{3.7}$$

$$\textit{where} \quad \mathcal{B} = \|\rho_0\|^2 + \|\rho_t(0)\|^2 + \|f\|^2_{L^\infty(I,L^2(\Omega))} + \|f_t\|^2_{L^\infty(I,L^2(\Omega))}. \tag{3.8}$$

Proof. Differentiate (3.1) in time to see that

$$\left\langle a|\mathbf{m}|^{-\alpha}\mathbf{m}_t - \alpha a|\mathbf{m}|^{-\alpha-1}\frac{\mathbf{m}\cdot\mathbf{m}_t}{|\mathbf{m}|}\mathbf{m} + a_t|\mathbf{m}|^{-\alpha}\mathbf{m}, \mathbf{v}\right\rangle - \langle \rho_t, \nabla\cdot\mathbf{v}\rangle = 0,$$
$$\langle \rho_{tt}, q\rangle + \langle \nabla\cdot\mathbf{m}_t, q\rangle = \langle f_t, q\rangle\,.$$

Taking $(\mathbf{v}, q) = (\mathbf{m}_t, \rho_t)$ and adding two resultant equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho_t\|^2 + \left\|a^{\frac{1}{2}}|\mathbf{m}|^{-\frac{\alpha}{2}}\mathbf{m}_t\right\|^2 = \alpha\left\langle a|\mathbf{m}|^{-\alpha-1}\frac{\mathbf{m}\cdot\mathbf{m}_t}{|\mathbf{m}|}\mathbf{m}, \mathbf{m}_t\right\rangle - \left\langle a_t|\mathbf{m}|^{-\alpha}\mathbf{m}, \mathbf{m}_t\right\rangle + \langle f_t, \rho_t\rangle\,. \tag{3.9}$$

Applying Hölder's inequality and Young's inequality, it follows that

$$\alpha\left\langle a|\mathbf{m}|^{-\alpha-1}\frac{\mathbf{m}\cdot\mathbf{m}_t}{|\mathbf{m}|}\mathbf{m}, \mathbf{m}_t\right\rangle - \left\langle a_t|\mathbf{m}|^{-\alpha}\mathbf{m}, \mathbf{m}_t\right\rangle \le \alpha\left\|a^{\frac{1}{2}}|\mathbf{m}|^{-\frac{\alpha}{2}}\mathbf{m}_t\right\|^2 + b^*\left\langle |\mathbf{m}|^{-\alpha}|\mathbf{m}|, |\mathbf{m}_t|\right\rangle$$
$$\le (\alpha+\varepsilon)\left\|a^{\frac{1}{2}}|\mathbf{m}|^{-\frac{\alpha}{2}}\mathbf{m}_t\right\|^2 + (b^*)^2 a_*^{-1}\varepsilon^{-1}\|\mathbf{m}\|_{0,s}^s\,, \tag{3.10}$$

and

$$\langle f_t, \rho_t\rangle \le \frac{1}{2}\|f_t\|^2 + \frac{1}{2}\|\rho_t\|^2\,. \tag{3.11}$$

Inserting (3.10)–(3.11) into (3.9) and taking $\varepsilon = (1-\alpha)/2$ yield

$$\frac{d}{dt}\|\rho_t\|^2 + (1-\alpha)\left\|a^{\frac{1}{2}}|\mathbf{m}|^{-\frac{\alpha}{2}}\mathbf{m}_t\right\|^2 \le \|f_t\|^2 + \|\rho_t\|^2 + C\|\mathbf{m}\|_{0,s}^s\,,$$

where $C = (b^*)^2(a_*(1-\alpha)/2)^{-1}$.

Neglecting the nonegative term $(1-\alpha)\left\|a^{\frac{1}{2}}|\mathbf{m}|^{-\frac{\alpha}{2}}\mathbf{m}_t\right\|^2$, and applying Grönwall's inequality, we find that

$$\|\rho_t\|_{L^\infty(I,L^2)}^2 \le C\left(\|\rho_t(0)\|^2 + \|f_t\|_{L^\infty(I,L^2)}^2 + \|\mathbf{m}\|_{L^s(I,L^s)}^s\right).$$

Combining this fact with estimate (3.2), we obtain the first part of estimate (3.7).

To verify the last part of (3.7), we choose $q = \nabla\cdot\mathbf{m}$ in (3.1), yielding

$$\|\nabla\cdot\mathbf{m}\|^2 = -\langle \rho_t, \nabla\cdot\mathbf{m}\rangle + \langle f, \nabla\cdot\mathbf{m}\rangle\,.$$

Then, by the Young inequality,

$$\|\nabla\cdot\mathbf{m}\|^2 \le 2(\|\rho_t\|^2 + \|f\|^2).$$

Using the first part of (3.7) to bound $\|\rho_t\|^2$, we find that $\|\nabla\cdot\mathbf{m}\|^2$ holds (3.7).

To verify the second part of (3.7), we take $(\mathbf{v}, q) = (\mathbf{m}, \rho)$ in (3.1), and adding the resultant equations yield

$$\langle A(\mathbf{m}), \mathbf{m}\rangle = -\langle \rho_t, \rho\rangle + \langle f, \rho\rangle\,. \tag{3.12}$$

By (2.6) and the boundedness of the function $a(\mathbf{x}, t)$ shows that

$$\langle A(\mathbf{m}), \mathbf{m}\rangle \ge a_*\|\mathbf{m}\|_{0,s}^s\,. \tag{3.13}$$

By the Young inequality,

$$-\langle \rho_t, \rho\rangle + \langle f, \rho\rangle \le \|\rho_t\|^2 + \|\rho\|^2 + \|f\|^2\,. \tag{3.14}$$

Combining (3.13)–(3.14) leads to

$$\|\mathbf{m}\|_{0,s}^s \le a_*^{-1}(\|\rho_t\|^2 + \|\rho\|^2 + \|f\|^2).$$

Using estimates (3.2) and (3.7) gives

$$\begin{aligned}\|\mathbf{m}\|^s_{L^\infty(I,L^s)} &\le C(\|\rho_t\|^2 + \|\rho\|^2 + \|f\|^2) \\ &\le C(\|\rho_0\|^2 + \|\rho_t(0)\|^2 + \|f\|^2_{L^\infty(I,L^2)} + \|f_t\|^2_{L^\infty(I,L^2)}),\end{aligned} \tag{3.15}$$

which completes the proof. □

4. The semidiscrete problem and error analysis

We assume that the boundary $\partial\Omega$ of Ω is polygonal or polyhedral. Let $\{\mathcal{T}_h\}_h$ be a regular triangulation of $\overline{\Omega}$ with $\max_{\tau\in\mathcal{T}_h} \operatorname{diam} \tau \le \mathrm{h}$. The discrete subspaces $V_h \times Q_h \subset V \times Q$ are defined as

$$Q_h = \{\rho_h \in L^2(\Omega), \forall\tau \in \mathcal{T}_h, \rho_h|_\tau \in P_0(\tau)\},$$
$$V_h = \{\mathbf{m}_h \in V, \forall\tau \in \mathcal{T}_h, \mathbf{m}_h|_\tau \in RT_0(\tau)\},$$

with $P_0(\tau)$ denoting the space of constants and

$$RT_0(\tau) = (P_0(\tau))^d + \mathbf{x}P_0(\tau).$$

So Q_h denotes the space of piecewise constant functions, while V_h is the lowest degree Raviart–Thomas space, (cf. Brezzi and Fortin (1991); Johnson and Thomée (1981); Bramble et al. (2002)). In what follows, we make use of the standard L^2-projection operator, see Ciarlet (1978), $\pi : Q \to Q_h$, satisfying

$$\begin{aligned}\langle \pi\rho - \rho, q\rangle = 0, &\quad \text{for all } \rho \in Q, q \in Q_h, \\ \langle \pi\rho - \rho, \nabla\cdot\mathbf{m}_h\rangle = 0, &\quad \text{for all } \mathbf{m}_h \in V_h, \rho \in Q.\end{aligned} \tag{4.1}$$

Furthermore, a projector Π can be defined on V mapping into V_h such that

$$\Pi : V \to V_h, \quad \langle \nabla\cdot(\Pi\mathbf{m} - \mathbf{m}), q\rangle = 0, \quad \text{for all } \mathbf{m} \in V, q \in Q_h. \tag{4.2}$$

These projections have well-known approximation properties, e.g. Brezzi and Fortin (1991); Johnson and Thomée (1981); Bramble et al. (2002).

$$\|\Pi\mathbf{m}\|_{0,q} \le C\left(\|\mathbf{m}\|_{0,q} + h\|\nabla\cdot\mathbf{m}\|\right), \quad \forall\mathbf{m} \in V \cap (W^{1,q}(\Omega))^d. \tag{4.3}$$

$$\|\Pi\mathbf{m} - \mathbf{m}\|_{0,q} \le Ch\|\mathbf{m}\|_{1,q}, \quad \forall\mathbf{m} \in V \cap (W^{1,q}(\Omega))^d. \tag{4.4}$$

$$\|\pi\rho\| \le C\|\rho\|, \quad \forall\rho \in L^2(\Omega). \tag{4.5}$$

$$\|\pi\rho - \rho\|_{0,q} \le Ch\|\rho\|_{1,q}, \quad q \in [1,\infty], \forall\rho \in W^{1,q}(\Omega). \tag{4.6}$$

The two projections π and Π preserve the commutative property $\operatorname{div} \circ \Pi = \pi \circ \operatorname{div} : V \to Q_h$. Our finite element approximation of the problem (3.1) is defined as follows: Find a pair $(\mathbf{m}_h, \rho_h) : I \to V_h \times Q_h$ such that

$$\begin{aligned}\langle A(\mathbf{m}_h), \mathbf{v}\rangle - \langle \rho_h, \nabla\cdot\mathbf{v}\rangle = 0 &\quad \text{for all } \mathbf{v} \in V_h, \\ \langle \rho_{ht}, q\rangle + \langle \nabla\cdot\mathbf{m}_h, q\rangle = \langle f, q\rangle &\quad \text{for all } q \in Q_h\end{aligned} \tag{4.7}$$

with initial data $\rho_h^0 = \pi\rho(\mathbf{x}, 0)$.

In the same manner to problem (3.1), we have the following:

Theorem 4.1. *Suppose* $(\mathbf{m}_h, \rho_h)$ *be a solution of the problem* (4.7)*. Then, there exists a positive constant* C *independence of* h *such that*

$$(i) \quad \|\rho_h\|^2_{L^\infty(I,L^2(\Omega))} + \|\mathbf{m}_h\|^s_{L^s(I,L^s(\Omega))} \le C\mathcal{A}, \tag{4.8}$$

$$(ii) \quad \|\rho_{ht}\|^2_{L^\infty(I,L^2(\Omega))} + \|\mathbf{m}_h\|^s_{L^\infty(I,L^s(\Omega))} + \|\nabla\cdot\mathbf{m}_h\|^2_{L^\infty(I,L^2(\Omega))} \le C\mathcal{B}, \tag{4.9}$$

where $\mathcal{A}$ *and* $\mathcal{B}$ *are defined as* (3.3) *and* (3.8)*, respectively.*

Proposition 4.2. *We have for all* $\mathbf{u}, \mathbf{v} \in V$,

$$\left(\|\mathbf{u}\|_{0,s} + \|\mathbf{v}\|_{0,s}\right)^\alpha \langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle \ge a_*(1-\alpha)\|\mathbf{u}-\mathbf{v}\|^2_{0,s}. \tag{4.10}$$

Proof. We have from (2.6) that

$$\begin{aligned}(1-\alpha)^{s/2}\|\mathbf{u}-\mathbf{v}\|^s_{0,s} &= \int_\Omega ((1-\alpha)|\mathbf{u}-\mathbf{v}|^2)^{s/2}dx\\ &\le \int_\Omega ((|\mathbf{u}|^{-\alpha}\mathbf{u} - |\mathbf{v}|^{-\alpha}\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}))^{s/2}(|\mathbf{u}|+|\mathbf{v}|)^{\alpha s/2}dx.\end{aligned}$$

Note that $(|\mathbf{u}|+|\mathbf{v}|)^{\alpha s/2} \in L^{2/\alpha}(\Omega)$ and $(|\mathbf{u}|^{-\alpha}\mathbf{u} - |\mathbf{v}|^{-\alpha}\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}))^{s/2} \in L^{2/s}(\Omega)$.

By Hölder's inequality,

$$\begin{aligned}(1-\alpha)^{s/2}\|\mathbf{u}-\mathbf{v}\|^s_{0,s} &\le \left\langle |\mathbf{u}|^{-\alpha}\mathbf{u} - |\mathbf{v}|^{-\alpha}\mathbf{v}, \mathbf{u}-\mathbf{v}\right\rangle^{s/2} \||\mathbf{u}|+|\mathbf{v}|\|^{\alpha s/2}_{0,s}\\ &\le a_*^{-s/2}\langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle^{s/2}\||\mathbf{u}|+|\mathbf{v}|\|^{\alpha s/2}_{0,s},\end{aligned}$$

or

$$\begin{aligned}a_*(1-\alpha)\|\mathbf{u}-\mathbf{v}\|^2_{0,s} &\le \langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle \||\mathbf{u}|+|\mathbf{v}|\|^\alpha_{0,s}\\ &\le \langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle \left(\|\mathbf{u}\|_{0,s} + \|\mathbf{v}\|_{0,s}\right)^\alpha.\end{aligned}$$

The proof is complete. □

Theorem 4.3. *There is a unique solution of the problem* (4.7) *satisfying* (4.8) *and* (4.9)*.*

Proof. Equation (4.7) can be interpreted as the finite system of ordinary differential equations in the coefficients of $(\mathbf{m}_h, \rho_h)$ with respect to the basis of $V_h \times Q_h$. The stability estimates (4.8) suffice to establish the local existence of $(\mathbf{m}_h(t), \rho_h(t))$ for all $t \in I$. The proof of this statement is essentially identical to that of Park (2005); Kim et al. (1996) for generalized Forchheimer flows. We will omit this.

Assume that $(\mathbf{m}_h^{(i)}, \rho_h^{(i)})$, $i = 1, 2$ are two solutions of (4.7). Let $\mathbf{m}_h = \mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}, \rho_h = \rho_h^{(1)} - \rho_h^{(2)}$. Then,

$$\begin{aligned}\left\langle A(\mathbf{m}_h^{(1)}) - A(\mathbf{m}_h^{(2)}), \mathbf{v}\right\rangle - \langle \rho_h, \nabla\cdot\mathbf{v}\rangle &= 0 \quad \text{for all } \mathbf{v}\in V_h,\\ \langle \rho_{ht}, q\rangle + \langle\nabla\cdot\mathbf{m}_h, q\rangle &= 0 \quad \text{for all } q \in Q_h.\end{aligned} \tag{4.11}$$

It is easy to see that with $\mathbf{v} = \mathbf{m}_h$ and $q = \rho_h$ in (4.11), adding the two resultant equations, one has

$$\frac{1}{2}\frac{d}{dt}\|\rho_h\|^2 + \left\langle A(\mathbf{m}_h^{(1)}) - A(\mathbf{m}_h^{(2)}), \mathbf{m}_h\right\rangle = 0.$$

Thanks to the monotonicity (2.6), we see that

$$\|\rho_h\|^2 + \left\langle A(\mathbf{m}_h^{(1)}) - A(\mathbf{m}_h^{(2)}), \mathbf{m}_h\right\rangle = \|\rho_h(0)\|^2 = 0.$$

Hence, $\rho_h = 0$ and $\left\langle A(\mathbf{m}_h^{(1)}) - A(\mathbf{m}_h^{(2)}), \mathbf{m}_h \right\rangle = 0$ a.e.

Due to (4.10), the boundedness of the functions $a(\mathbf{x}, t)$, $\left\|\mathbf{m}_h^{(1)}\right\|_{0,s}$ and $\left\|\mathbf{m}_h^{(2)}\right\|_{0,s}$, we have

$$\|\mathbf{m}_h\|_{0,s}^2 \le a_*^{-1}(1-\alpha)^{-1} \left\langle A(\mathbf{m}_h^{(1)}) - A(\mathbf{m}_h^{(2)}), \mathbf{m}_h \right\rangle \left(\left\|\mathbf{m}_h^{(1)}\right\|_{0,s} + \left\|\mathbf{m}_h^{(2)}\right\|_{0,s} \right)^\alpha = 0.$$

Thus, $\mathbf{m}_h = 0$ a.e. □

4.1. Error estimates

In this subsection, we will give the error estimate between the analytical solution and approximate solution. We define the new variables:

$$\mathbf{m} - \mathbf{m}_h = \mathbf{m} - \Pi\mathbf{m} - (\mathbf{m}_h - \Pi\mathbf{m}) = \eta - \zeta_h,$$
$$\rho - \rho_h = \rho - \pi\rho - (\rho_h - \pi\rho) = \theta - \vartheta_h.$$

Theorem 4.4. *Let* $(\mathbf{m}, \rho)$ *be the solution of* (3.1) *and* $(\mathbf{m}_h, \rho_h)$ *be the solution of* (4.7). *Suppose that* $(\mathbf{m}, \rho) \in V \times Q$, *and* $\rho_t \in L^2(I, L^2(\Omega))$. *Then, there exists a positive constant* C *independent of* h *such that*

$$\|\rho - \rho_h\|_{L^\infty(I,L^2(\Omega))}^2 + \int_0^T \|\mathbf{m} - \mathbf{m}_h\|_{0,s}^2 \, dt \le Ch^{2(1-\alpha)}. \tag{4.12}$$

Proof. By (3.1) and (4.7), we have the error equations

$$\begin{aligned} \langle A(\mathbf{m}) - A(\mathbf{m}_h), \mathbf{v} \rangle - \langle \rho - \rho_h, \nabla \cdot \mathbf{v} \rangle &= 0 \quad \text{for all } \mathbf{v} \in V_h, \\ \langle \rho_t - \rho_{ht}, q \rangle + \langle \nabla \cdot (\mathbf{m} - \mathbf{m}_h), q \rangle &= 0 \quad \text{for all } q \in Q_h. \end{aligned} \tag{4.13}$$

Using L^2-project(4.1) and the Raviar–Thomas projection (4.2), we rewrite (4.13) as form

$$\langle A(\mathbf{m}) - A(\Pi\mathbf{m}), \mathbf{v} \rangle + \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \mathbf{v} \rangle + \langle \vartheta_h, \nabla \cdot \mathbf{v} \rangle = 0 \quad \text{for all } \mathbf{v} \in V_h$$
$$\langle \theta_t, q \rangle - \langle \vartheta_{ht}, q \rangle - \langle \nabla \cdot \zeta_h, q \rangle = 0 \quad \text{for all } q \in Q_h.$$

Taking $q = -\vartheta_h \in Q_h$ and $\mathbf{v} = -\zeta_h \in V_h$, and adding these two equations together, we get

$$\frac{1}{2}\frac{d}{dt} \|\vartheta_h\|^2 + \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h \rangle = \langle A(\mathbf{m}) - A(\Pi\mathbf{m}), \zeta_h \rangle + \langle \theta_t, \vartheta_h \rangle .$$

Using Young's and Hölder's inequality, we find that

$$\langle \theta_t, \vartheta_h \rangle \le \frac{1}{2\varepsilon} \|\theta_t\|^2 + \frac{\varepsilon}{2} \|\vartheta_h\|^2 ,$$

and by (2.5) with note that $(1-\alpha)s^* = s$, applying Hölder's inequality and Young's inequality shows that

$$\begin{aligned} \langle A(\mathbf{m}) - A(\Pi\mathbf{m}), \zeta_h \rangle &\le 2 \left\langle |\eta|^{1-\alpha}, |\zeta_h| \right\rangle \le 2 \|\eta\|_{0,s}^{1-\alpha} \|\zeta_h\|_{0,s} \\ &\le \frac{1}{2} a_*(1-\alpha)(\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s})^{-\alpha} \|\zeta_h\|_{0,s}^2 \\ &\quad + 2(a_*(1-\alpha))^{-1}(\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s})^{\alpha} \|\eta\|_{0,s}^{2(1-\alpha)} . \end{aligned}$$

Then, by Proposition 4.10,

$$\langle A(\mathbf{m}) - A(\Pi\mathbf{m}), \zeta_h\rangle \le \frac{1}{2}\langle A(|\Pi\mathbf{m}|) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle + 2(a_*(1-\alpha))^{-1}(\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s})^\alpha \|\eta\|_{0,s}^{2(1-\alpha)}. \tag{4.14}$$

We find that

$$\frac{d}{dt}\|\vartheta_h\|^2 + \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle \le \varepsilon\|\vartheta_h\|^2 + 4(a_*(1-\alpha))^{-1}\Big(\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s}\Big)^\alpha \|\eta\|_{0,s}^{2(1-\alpha)} + \varepsilon^{-1}\|\theta_t\|^2.$$

Integrating in time from 0 to t and taking the sup-norm,

$$\sup_{t\in[0,T]}\|\vartheta_h\|^2 + \int_0^T \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle\, dt \le \varepsilon T \sup_{t\in[0,T]}\|\vartheta_h\|^2 + \varepsilon^{-1}\int_0^T \|\theta_t\|^2 dt + 4(a_*(1-\alpha))^{-1}\int_0^T (\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s})^\alpha \|\eta\|_{0,s}^{2(1-\alpha)} dt.$$

Now taking $\varepsilon = 1/(2T)$, we find that

$$\sup_{t\in[0,T]}\|\vartheta_h\|^2 + 2\int_0^T \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle\, dt \le (4T + 8(a_*(1-\alpha))^{-1})\Big(\int_0^T \|\theta_t\|^2 dt + \int_0^T (\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s})^\alpha \|\eta\|_{0,s}^{2(1-\alpha)} dt\Big).$$

Thus,

$$\|\vartheta_h\|^2_{L^\infty(I,L^2)} + 2\int_0^T \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle\, dt \le (4T + 8(a_*(1-\alpha))^{-1})\Big(\int_0^T \|\theta_t\|^2 dt + \int_0^T (\|\Pi\mathbf{m}\|_{0,s} + \|\mathbf{m}_h\|_{0,s})^\alpha \|\eta\|_{0,s}^{2(1-\alpha)} dt\Big).$$

Then, by (4.3), (3.7) and (4.9),

$$\|\vartheta_h\|^2_{L^\infty(I,L^2)} + 2\int_0^T \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle\, dt \le C_1\Big(\int_0^T \|\theta_t\|^2 d\tau + \int_0^T \|\eta\|_{0,s}^{2(1-\alpha)} d\tau\Big), \tag{4.15}$$

where $C_1 = C2^\alpha \mathcal{B}^{\alpha/s}(4T + 8(a_*(1-\alpha))^{-1})$.

Dropping the nonnegative term $\int_0^T \langle A(\Pi\mathbf{m}) - A(\mathbf{m}_h), \Pi\mathbf{m} - \mathbf{m}_h\rangle\, dt$, we find that

$$\|\vartheta_h\|^2_{L^\infty(I,L^2)} \le C_1\Big(\int_0^T \|\theta_t\|^2 d\tau + \int_0^T \|\eta\|_{0,s}^{2(1-\alpha)} d\tau\Big). \tag{4.16}$$

Using (2.3) and (4.16), we obtain

$$\|\rho - \rho_h\|^2_{L^\infty(I,L^2)} \le 2\left(\|\theta\|^2_{L^\infty(I,L^2)} + \|\vartheta_h\|^2_{L^\infty(I,L^2)}\right) \le 2C_1\Big(\|\theta\|^2_{L^\infty(I,L^2)} + \int_0^T \|\theta_t\|^2 d\tau + \int_0^T \|\eta\|_{0,s}^{2(1-\alpha)} d\tau\Big).$$

Applying estimates (4.4) and (4.6) imply that

$$\|\rho-\rho_h\|^2_{L^\infty(I,L^2)} \le Ch^2\Big(\|\rho\|^2_{1,2}+\int_0^T\|\rho_t\|^2_{1,2}\,d\tau\Big)+Ch^{2(1-\alpha)}\int_0^T\|\mathbf{m}\|^{2(1-\alpha)}_{1,s}\,d\tau. \tag{4.17}$$

By (4.10) and the triangle inequality,

$$\begin{aligned}\int_0^T\|\mathbf{m}-\mathbf{m}_h\|^2_{0,s}\,d\tau &\le 2\int_0^T\|\eta\|^2_{0,s}+\|\zeta_h\|^2_{0,s}\,d\tau\\ &\le C_2\Big(\int_0^T\|\eta\|^2_{0,s}\,d\tau+\int_0^T(\|\Pi\mathbf{m}\|_{0,s}+\|\mathbf{m}_h\|_{0,s})^\alpha\langle A(\Pi\mathbf{m})-A(\mathbf{m}_h),\Pi\mathbf{m}-\mathbf{m}_h\rangle\,dt\Big)\\ &\le C_3\Big(\int_0^T\|\eta\|^2_{0,s}\,dt+\int_0^T\langle A(\Pi\mathbf{m})-A(\mathbf{m}_h),\Pi\mathbf{m}-\mathbf{m}_h\rangle\,dt\Big),\end{aligned}$$

where $C_2=2+(a_*(1-\alpha))^{-1}$, $C_3=CC_2(2^\alpha\mathcal{B}^{\alpha/s}+1)$.
According to (4.15), we get

$$\begin{aligned}\int_0^T\|\mathbf{m}-\mathbf{m}_h\|^2_{0,s}\,dt &\le C_1C_3\Big(\int_0^T\|\eta\|^2_{0,s}\,d\tau+\int_0^T\|\theta_t\|^2\,d\tau+\int_0^T\|\eta\|^{2(1-\alpha)}_{0,s}\,d\tau\Big)\\ &\le Ch^2\Big(\int_0^T\|\mathbf{m}\|^2_{1,s}\,d\tau+\int_0^T\|\rho_t\|^2_{1,2}\,d\tau\Big)+Ch^{2(1-\alpha)}\int_0^T\|\mathbf{m}\|^{2(1-\alpha)}_{1,s}\,d\tau.\end{aligned} \tag{4.18}$$

The result follows directly from (4.17) and (4.18), concluding the proof. □

5. The fully discrete mixed discretization and error analysis

We now proceed with time discretization for problem (4.7), which is achieved by the backward Euler scheme. Let $N\ge 1$ give the time step $\tau=T/N$. For a given $n=1,2,\dots,N$, with $t_n=n\tau$. For any function φ of time, we denote $\varphi^n=\varphi(\cdot,t_n)$. We also use the notation $A(\varphi^n)$ in place of $A(\cdot,t_n,\varphi^n)$. The discrete time mixed finite element approximation to (4.7) is defined as follows: For given $\rho_h^0(\mathbf{x})=\pi\rho_0(\mathbf{x})$ and $\{f^n\}_{n=1}^N\in L^2(\Omega)$. Find a pair $(\mathbf{m}_h^n,\rho_h^n)$ in $V_h\times Q_h$, $n=0,1,2,\dots,N$ such that

$$\begin{aligned}\langle A(\mathbf{m}_h^n),\mathbf{v}\rangle-\langle\rho_h^n,\nabla\cdot\mathbf{v}\rangle=0 &\quad \text{for all } \mathbf{v}\in V_h,\\ \Big\langle\frac{\rho_h^n-\rho_h^{n-1}}{\tau},q\Big\rangle+\langle\nabla\cdot\mathbf{m}_h^n,q\rangle=\langle f^n,q\rangle &\quad \text{for all } q\in Q_h.\end{aligned} \tag{5.1}$$

Lemma 5.1 (Stability)**.** *Let* $(\mathbf{m}_h^n,\rho_h^n)$ *solve the fully discrete finite element approximation* (5.1) *for each time step* $n=1,2,\dots,N$. *There exists a positive constant* C *independent of* n,τ *such that for a* τ *sufficiently small*

$$\|\rho_h^n\|^2+\sum_{i=1}^n\tau\left\|\mathbf{m}_h^i\right\|^s_{0,s}\le C\Big(\|\rho_0\|^2+\sum_{i=1}^n\tau\left\|f^i\right\|^2\Big). \tag{5.2}$$

Proof. Selecting $(\mathbf{v},q)=2(\mathbf{m}_h^n,\rho_h^n)$ in (5.1), we find that

$$\begin{aligned}2\langle A(\mathbf{m}_h^n),\mathbf{m}_h^n\rangle-2\langle\rho_h^n,\nabla\cdot\mathbf{m}_h^n\rangle&=0,\\ 2\Big\langle\frac{\rho_h^n-\rho_h^{n-1}}{\tau},\rho_h^n\Big\rangle+2\langle\nabla\cdot\mathbf{m}_h^n,\rho_h^n\rangle&=2\langle f^n,\rho_h^n\rangle.\end{aligned} \tag{5.3}$$

Adding the two above equations, and using the identity,

$$2\left\langle \frac{\rho_h^n-\rho_h^{n-1}}{\tau},\rho_h^n\right\rangle=\frac{1}{\tau}(\|\rho_h^n\|^2-\left\|\rho_h^{n-1}\right\|^2+\left\|\rho_h^n-\rho_h^{n-1}\right\|^2),$$

we obtain

$$\frac{1}{\tau}(\|\rho_h^n\|^2-\left\|\rho_h^{n-1}\right\|^2+\left\|\rho_h^n-\rho_h^{n-1}\right\|^2)+2\left\langle A(\mathbf{m}_h^n),\mathbf{m}_h^n\right\rangle=2\left\langle f^n,\rho_h^n\right\rangle. \tag{5.4}$$

It follows from (2.6) that

$$\left\langle A(\mathbf{m}_h^n),\mathbf{m}_h^n\right\rangle\geq a_*(1-\alpha)\left\|\mathbf{m}_h^n\right\|_{0,s}^s. \tag{5.5}$$

Using Hölder's inequality to the RHS of (5.4) shows that

$$2\left\langle f^n,\rho_h^n\right\rangle\leq\|f^n\|^2+\|\rho_h^n\|^2. \tag{5.6}$$

Combining (5.4)–(5.6) and dropping the nonnegative term $\left\|\rho_h^n-\rho_h^{n-1}\right\|^2$ yields

$$\frac{\|\rho_h^n\|^2-\left\|\rho_h^{n-1}\right\|^2}{\tau}-\|\rho_h^n\|^2+a_*(1-\alpha)\left\|\mathbf{m}_h^n\right\|_{0,s}^s\leq C\|f^n\|^2.$$

By the discrete Grönwall inequality in Lemma 2.2,

$$\|\rho_h^n\|^2+C_3\sum_{i=1}^n\tau\left\|\mathbf{m}_h^i\right\|_{0,s}^s\leq Ce^{\frac{n\tau}{1-\tau}}\left\|\rho_h^0\right\|^2+Ce^{\frac{n\tau}{1-\tau}}\sum_{i=1}^n\tau\left\|f^i\right\|^2. \tag{5.7}$$

Note that $\|\rho_h^0\|^2\leq\|\rho_0\|^2$ and $e^{\frac{n\tau}{1-\tau}}\leq e^{\frac{N\tau}{1-\tau}}=e^{\frac{T}{1-\tau}}$ imply (5.2). The proof is complete. □

5.1. Error analysis

As in the semidiscrete case, we use $\eta=\mathbf{m}-\Pi\mathbf{m}$, $\zeta_h=\mathbf{m}_h-\Pi\mathbf{m}$, $\theta=\rho-\pi\rho$, $\vartheta_h=\rho_h-\pi\rho$ and η^n,θ^n, ζ_h^n,ϑ_h^n by evaluating η,θ, ζ_h,ϑ_h at the discrete time levels. We also define

$$\partial\varphi^n=\frac{\varphi^n-\varphi^{n-1}}{\tau}.$$

First, we establish some results that are crucial in getting the convergent results.

Lemma 5.2. *For $n\geq 1$ if $\rho_t,\rho_{tt}\in L^2(0,T;L^2(\Omega))$, then*

$$(i)\quad \|\partial\rho^n\|^2\leq\tau^{-1}\int_{t_{n-1}}^{t_n}\|\rho_t\|^2\,dt. \tag{5.8}$$

$$(ii)\quad \|\rho_t^n-\partial\rho^n\|^2\leq\frac{\tau}{3}\int_{t_{n-1}}^{t_n}\|\rho_{tt}\|^2\,dt. \tag{5.9}$$

Proof. Proof of (i) By the Fundamental Theorem of Calculus, we have

$$\frac{\rho^n-\rho^{n-1}}{\tau}=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\rho_t dt.$$

By means of Hölder's inequality, we find that

$$\begin{aligned}\|\partial\rho^n\|^2 &= \tau^{-2}\int_\Omega\left(\int_{t_{n-1}}^{t_n}\rho_t dt\right)^2 dx\\ &\le \tau^{-2}\int_\Omega(\int_{t_{n-1}}^{t_n} dt)(\int_{t_{n-1}}^{t_n}|\rho_t|^2 dt)dx \quad \text{(Hölder's inequality)}\\ &= \tau^{-1}\int_{t_{n-1}}^{t_n}\|\rho_t\|^2\,dt.\end{aligned}$$

This proves (5.8).

Proof of (ii) By Taylor expansion with integral remainder,

$$\rho^{n-1} = \rho^n - \tau\rho_t^n + \int_{t_{n-1}}^{t_n}\rho_{tt}(t)(t-t_{n-1})dt.$$

This implies that

$$\|\rho_t^n - \partial\rho^n\|^2 = \tau^{-2}\int_\Omega\Big|\int_{t_{n-1}}^{t_n}\rho_{tt}(t)(t_n-t)dt\Big|^2 dx. \tag{5.10}$$

We estimate the right hand side of (5.10) by Hölder's inequality

$$\begin{aligned}\int_\Omega\Big|\int_{t_{n-1}}^{t_n}\rho_{tt}(t)(t_n-t)dt\Big|^2 &\le \int_\Omega\Big(\int_{t_{n-1}}^{t_n}|\rho_{tt}|^2dt\int_{t_{n-1}}^{t_n}(t_n-t)^2dt\Big)dx\\ &\le \frac{\tau^3}{3}\Big(\int_\Omega\int_{t_{n-1}}^{t_n}|\rho_{tt}|^2dtdx\Big) = \frac{\tau^3}{3}\int_{t_{n-1}}^{t_n}\|\rho_{tt}\|^2\,dt.\end{aligned} \tag{5.11}$$

Then, (5.9) follows directly from inserting (5.11) into (5.10). □

Theorem 5.3. *Let* $(\mathbf{m}^n, \rho^n)$ *solve problem* (3.1) *and* $(\mathbf{m}_h^n, \rho_h^n)$ *solve the fully discrete finite element approximation* (5.1) *for each time step* n*,* $n = 1, \ldots, N$*. Suppose that* $(\mathbf{m}, \rho) \in V \times Q$ *and* $\rho_{tt} \in L^2(I, L^2(\Omega))$*. Then, there exists a positive constant* C *independent of* h *and* τ *such that for a sufficiently small* τ*,*

$$\|\rho^n - \rho_h^n\|^2 + \|\mathbf{m}^n - \mathbf{m}_h^n\|_{0,s}^s \le C\big(h^{2(1-\alpha)} + \tau^2\big). \tag{5.12}$$

Proof. Evaluating equation (3.1) at $t = t_n$ gives

$$\begin{aligned}\langle A(\mathbf{m}^n), \mathbf{v}\rangle - \langle\rho^n, \nabla\cdot\mathbf{v}\rangle &= 0 \quad \text{for all } \mathbf{v}\in V_h,\\ \langle\rho_t^n, q\rangle + \langle\nabla\cdot\mathbf{m}^n, q\rangle &= \langle f^n, q\rangle \quad \text{for all } q\in Q_h.\end{aligned} \tag{5.13}$$

Subtracting (5.1) from (5.13), we obtain

$$\langle A(\mathbf{m}^n) - A(\mathbf{m}_h^n), \mathbf{v}\rangle - \langle\pi\rho^n - \rho_h^n, \nabla\cdot\mathbf{v}\rangle = 0, \quad \text{for all } \mathbf{v}\in V_h, \tag{5.14}$$

$$\left\langle\rho_t^n - \frac{\rho_h^n - \rho_h^{n-1}}{\tau}, q\right\rangle + \langle\nabla\cdot(\Pi\mathbf{m}^n - \mathbf{m}_h^n), q\rangle = 0 \quad \text{for all } q\in Q_h. \tag{5.15}$$

Choosing $\mathbf{v} = -\zeta_h^n, q = -\vartheta_h^n$, and adding the two equations, we obtain

$$\langle\rho_t^n - \partial\rho_h^n, \vartheta_h^n\rangle - \langle A(\mathbf{m}^n) - A(\mathbf{m}_h^n), \Pi\mathbf{m}^n - \mathbf{m}_h^n\rangle = 0. \tag{5.16}$$

Since $\rho_t^n - \partial\rho_h^n = \rho_t^n - \partial\rho^n + \partial\theta^n - \partial\vartheta_h^n$, we rewrite (5.16) in the form

$$\begin{aligned}&\langle\partial\vartheta_h^n, \vartheta_h^n\rangle + \langle A(\Pi\mathbf{m}^n) - A(\mathbf{m}_h^n), \Pi\mathbf{m}^n - \mathbf{m}_h^n\rangle\\ &\quad = \langle A(\mathbf{m}^n) - A(\Pi\mathbf{m}^n), \zeta_h^n\rangle + \langle\rho_t^n - \partial\rho^n, \vartheta_h^n\rangle + \langle\partial\theta^n, \vartheta_h^n\rangle\,.\end{aligned} \tag{5.17}$$

We estimate (5.17) term by term.
For the first term, we use the identity

$$\langle \partial \vartheta_h^n, \vartheta_h^n \rangle = \frac{\|\vartheta_h^n\|^2 - \left\|\vartheta_h^{n-1}\right\|^2}{2\tau} + \frac{\tau}{2}\left\|\partial \vartheta_h^n\right\|^2. \tag{5.18}$$

For the third term, by (2.5), using Hölder's ineqiality and the Young inequality gives

$$\begin{aligned}\langle A(\mathbf{m}^n) - A(\Pi\mathbf{m}^n), \zeta_h^n\rangle &\le \frac{a_*(1-\alpha)}{2}(\|\Pi\mathbf{m}^n\|_{0,s} + \|\mathbf{m}_h^n\|_{0,s})^{-\alpha}\|\zeta_h^n\|_{0,s}^2 \\ &\quad + 2(a_*(1-\alpha))^{-1}(\|\Pi\mathbf{m}^n\|_{0,s} + \|\mathbf{m}_h^n\|_{0,s})^{\alpha}\|\eta^n\|_{0,s}^{2(1-\alpha)} \\ &\le \frac{1}{2}\langle A(|\Pi\mathbf{m}^n|) - A(\mathbf{m}_h^n), \Pi\mathbf{m}^n - \mathbf{m}_h^n\rangle \\ &\quad + 2(a_*(1-\alpha))^{-1}(\|\Pi\mathbf{m}^n\|_{0,s} + \|\mathbf{m}_h^n\|_{0,s})^{\alpha}\|\eta^n\|_{0,s}^{2(1-\alpha)}.\end{aligned} \tag{5.19}$$

Using Young's inequalities, (5.8) and (5.9), we obtain

$$\begin{aligned}\langle \rho_t^n - \partial\rho^n, \vartheta_h^n\rangle + \langle \partial\theta^n, \vartheta_h^n\rangle &\le \|\rho_t^n - \partial\rho^n\|^2 + \frac{1}{2}\|\vartheta_h^n\|^2 + \|\partial\theta^n\|^2 \\ &\le \frac{\tau}{3}\int_{t_{n-1}}^{t_n}\|\rho_{tt}\|^2 d\tau + \tau^{-1}\int_{t_{n-1}}^{t_n}\|\theta_t\|^2 d\tau + \frac{1}{2}\|\vartheta_h^n\|^2.\end{aligned} \tag{5.20}$$

In view of (5.18)–(5.20), (5.17) yields

$$\begin{aligned}\frac{\|\vartheta_h^n\|^2 - \left\|\vartheta_h^{n-1}\right\|^2}{\tau} - \|\vartheta_h^n\|^2 + \langle A(\Pi\mathbf{m}^n) - A(\mathbf{m}_h^n), \Pi\mathbf{m}^n - \mathbf{m}_h^n\rangle \\ \le \frac{2}{3}\tau\int_{t_{n-1}}^{t_n}\|\rho_{tt}\|^2 d\tau + 2\tau^{-1}\int_{t_{n-1}}^{t_n}\|\theta_t\|^2 d\tau \\ + 4(a_*(1-\alpha))^{-1}(\|\Pi\mathbf{m}^n\|_{0,s} + \|\mathbf{m}_h^n\|_{0,s})^{\alpha}\|\eta^n\|_{0,s}^{2(1-\alpha)}.\end{aligned}$$

Using (4.3), (4.8) and (4.9), we find that

$$\begin{aligned}\frac{\|\vartheta_h^n\|^2 - \left\|\vartheta_h^{n-1}\right\|^2}{\tau} - \|\vartheta_h^n\|^2 + \langle A(\Pi\mathbf{m}^n) - A(\mathbf{m}_h^n), \Pi\mathbf{m}^n - \mathbf{m}_h^n\rangle \\ \le \tau\int_{t_{n-1}}^{t_n}\|\rho_{tt}\|^2 d\tau + 2\tau^{-1}\int_{t_{n-1}}^{t_n}\|\theta_t\|^2 d\tau + 4(a_*(1-\alpha))^{-1}2^{\alpha}\mathcal{B}^{\alpha/s}\|\eta^n\|_{0,s}^{2(1-\alpha)}.\end{aligned} \tag{5.21}$$

By means of the discrete form of Grönwall's inequality in Lemma 2.2 and the fact $\vartheta_h^0 = 0$, we find that

$$\begin{aligned}\|\vartheta_h^n\|^2 + \sum_{i=1}^{n}\tau\left\langle A(\Pi\mathbf{m}^i) - A(\mathbf{m}_h^i), \Pi\mathbf{m}^i - \mathbf{m}_h^i\right\rangle \\ \le C_1\Big(\tau^2\int_0^T\|\rho_{tt}\|^2 d\tau + \int_0^T\|\theta_t\|^2 d\tau + \tau\sum_{i=1}^{n}\left\|\eta^i\right\|_{0,s}^{2(1-\alpha)}\Big),\end{aligned} \tag{5.22}$$

where $C_1 = (2 + 4(a_*(1-\alpha))^{-1}2^{\alpha}\mathcal{B}^{\alpha/s})e^{\frac{T}{1-\tau}}$.

Using (2.3) and (4.10), we find that

$$\|\rho^n - \rho_h^n\|^2 + \sum_{i=1}^n \tau \left\|\mathbf{m}^i - \mathbf{m}_h^i\right\|_{0,s}^2 \le 2\Big(\|\theta^n\|^2 + \sum_{i=1}^n \tau \left\|\eta^i\right\|_{0,s}^2 + \|\vartheta_h^n\|^2 + \sum_{i=1}^n \tau \left\|\zeta_h^i\right\|_{0,s}^2 \Big)$$

$$\le (2 + (a_*(1-\alpha))^{-1})\Big(\|\theta^n\|^2 + \sum_{i=1}^n \tau \left\|\eta^i\right\|_{0,s}^2 + \|\vartheta_h^n\|^2$$

$$+ \sum_{i=1}^n \tau(\left\|\Pi\mathbf{m}^i\right\|_{0,s} + \left\|\mathbf{m}_h^i\right\|_{0,s})^\alpha \left\langle A(\Pi\mathbf{m}^i) - A(\mathbf{m}_h^i), \Pi\mathbf{m}^i - \mathbf{m}_h^i \right\rangle \Big)$$

$$\le C_2\Big(\|\theta^n\|^2 + \sum_{i=1}^n \tau \left\|\eta^i\right\|_{0,s}^2 + \|\vartheta_h^n\|^2 + \sum_{i=1}^n \tau \left\langle A(\Pi\mathbf{m}^i) - A(\mathbf{m}_h^i), \Pi\mathbf{m}^i - \mathbf{m}_h^i \right\rangle),$$

where $C_2 = C(2 + (a_*(1-\alpha))^{-1})(2^\alpha \mathcal{B}^{\alpha/s} + 1)$.

Then, by (5.22),

$$\|\rho^n - \rho_h^n\|^2 + \sum_{i=1}^n \tau \left\|\mathbf{m}^i - \mathbf{m}_h^i\right\|_{0,s}^2 \le (C_1+1)C_2\Big(\|\theta^n\|^2 + \sum_{i=1}^n \tau \left\|\eta^i\right\|_{0,s}^2$$

$$+ \tau^2 \int_0^T \|\rho_{tt}\|^2 \, d\tau + \int_0^T \|\theta_t\|^2 \, d\tau + \sum_{i=1}^N \tau \left\|\eta^i\right\|_{0,s}^{2(1-\alpha)} \Big).$$

Applying (4.4) and (4.6) gives

$$\|\rho^n - \rho_h^n\|^2 + \sum_{i=1}^n \tau \left\|\mathbf{m}^i - \mathbf{m}_h^i\right\|_{0,s}^2 \le C\Big(h^2 \|\rho^n\|_{1,2}^2 + h^2 \sum_{i=1}^n \tau \left\|\mathbf{m}^i\right\|_{1,s}^2$$

$$+ \tau^2 \int_0^T \|\rho_{tt}\|_{L^2}^2 \, d\tau + h^{2(1-\alpha)} \sum_{i=1}^n \tau \left\|\mathbf{m}^i\right\|_{1,s}^{2(1-\alpha)} \Big).$$

This completes the proof. □

6. Numerical results

In this section, we carry out numerical simulations using mixed finite element approximation to solve problem (5.1) in two dimensions to validate our theoretical estimates. For simplicity, the region of examples are unit square $\Omega = [0,1]^2$. We use the piecewise constant elements for the density variable and lowest order Raviart–Thomas for momentum variable. We divided the unit square into an $\mathcal{N} \times \mathcal{N}$ mesh of squares, each of them subdivided into two right triangles. The triangularization in the region Ω is a uniform subdivision in each dimension. The calculations are performed for $T = 2$. For each mesh, we solve the problem (5.1) numerically. Our problem is solved at each time level starting at $t = 0$ until the given final time T. The relative error control in each nonlinear solve is $\mathrm{tol} = 10^{-6}$. At time T, we measure the error in the L^2-norm for the density and the L^s -norm for the vector momentum. We obtain the convergence rates $r_i = \frac{\ln e_{i-1} - \ln e_i}{\ln h_{i-1} - \ln h_i}$ of finite approximation at seven levels with the discretization parameters $h \in \{1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$(the mesh size is actually $h\sqrt{2}$), respectively. In view of Theorem 5.3, the time step is taken $\Delta t = 0.5h^{1-\alpha}$ (equating the exponents in the error bound of (5.12)) to ensure that the terms τ^2 and $h^{(1-\alpha)}$ are of the same order. We compute the errors as given in (5.12). We test the convergence of our method with $\alpha = \frac{1}{2}$, $s = \frac{3}{2}$. To test the convergence rates, we choose the analytical solution

$$\rho(\mathbf{x}, t) = e^{-t} \sin(\pi x_1) \sin(\pi x_2).$$

$$\mathbf{m}(\mathbf{x}, t) = -(\pi e^{-t})^2 \sqrt{\cos^2(\pi x_1) \sin^2(\pi x_2) + \sin^2(\pi x_1) \cos^2(\pi x_2)} \begin{bmatrix} \cos(\pi x_1) \sin(\pi x_2) \\ \sin(\pi x_1) \cos(\pi x_2) \end{bmatrix}.$$

For simplicity, we take $a(\mathbf{x}, t) = 1$ on $\overline{\Omega} \times I$. The forcing term f is determined from equation $\rho_t + \nabla \cdot \mathbf{m} = f$. Explicitly,

$$f(\mathbf{x}, t) = -e^{-t} \sin(\pi x_1) \sin(\pi x_2) - 2\pi^3 e^{-2t} \frac{\sin(\pi x_1) \sin(\pi x_2)[2\cos(2\pi x_1)\cos(2\pi x_2) + \cos\pi(x_1 + x_2)\cos\pi(x_1 - x_2) - 1]}{\sqrt{2 - 2\cos(2\pi x_1)\cos(2\pi x_2)}}.$$

The initial condition and boundary condition are determined according to the analytical solution as follows:

$$\rho_0(\mathbf{x}) = \sin \pi x_1 \sin \pi x_2 \quad \mathbf{x} \in \Omega,$$
$$\rho(\mathbf{x}) = 0 \quad \text{on } \partial\Omega.$$

The numerical results are listed below in Table 6.1.

$\mathcal{N}$	h	τ	Error	$\tau^2 + h^{2(1-\alpha)}$	Conv. order
4	3.536E-01	2.973E-01	4.176E-02	4.419E-01	–
8	1.768E-01	2.102E-01	1.828E-02	2.210E-01	1.192
16	8.838E-02	1.487E-01	6.063E-03	1.105E-01	1.592
32	4.419E-02	1.051E-01	2.024E-03	5.524E-02	1.583
64	2.210E-02	7.433E-02	6.753E-04	2.762E-02	1.584
128	1.105E-02	5.256E-02	2.250E-04	1.381E-02	1.586
256	5.524E-03	3.176E-02	7.500E-05	6.905E-03	1.585

TABLE 6.1. *Numerical results (final time $T = 2$, $\tau = 0.5\sqrt{h}$).*

As shown in table 6.1, the numerical results confirm the theoretically estimated convergence order of $\tau^2 + h$.

7. Conclusions

In this paper, we have analysed a numerical scheme for slightly incomprehensible pre-Darcy flows. The spatial discretization is mixed and based on the lowest order Raviart–Thomas finite elements, whereas the time step is performed by the backward Euler method. We have proven the convergence of the scheme by estimating the error in terms of the discretization parameters. The numerical experiment agrees with the estimates derived theoretically.

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